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I.D. number.....

Math 2250-4
FINAL EXAM
December 17, 2013

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform Tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	_____	40
2	_____	15
3	_____	15
4	_____	15
5	_____	25
6	_____	10
7	_____	30
total	_____	150

1) Constant coefficient first order linear differential equations with constant right hand sides can be solved with most of the techniques we studied in this course. Find the general solution to the differential equation for $x = x(t)$ below, using the specified techniques.

$$x'(t) + 2x(t) = -8.$$

1a) Use the integrating factor technique for first order linear differential equations.

(10 points)

1b) Use the technique for separable differential equations.

(10 points)

1c) Use the technique based on superposition of particular and homogeneous solutions. For your convenience, the differential equation is repeated here:

$$x'(t) + 2x(t) = -8.$$

(10 points)

1d) Use Laplace transform. Note that in this case the free parameter in your solution will be the initial value x_0 .

(10 points)

2) Here is a matrix and its reduced row echelon form:

$$A := \begin{bmatrix} -2 & 2 & 2 & 4 & 1 \\ 3 & -1 & -5 & 0 & 0 \\ 0 & 1 & -1 & 3 & -3 \\ 3 & 2 & -8 & 9 & -2 \end{bmatrix}; \text{ reduced row echelon form of } A: \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2a) Find the solution space of vectors \underline{x} that solve the homogeneous matrix equation $A\underline{x} = \underline{0}$. Write your explicit solution (with free parameters), in linear combination form.

(10 points)

2b) It is easy to check that

$$\begin{bmatrix} -2 & 2 & 2 & 4 & 1 \\ 3 & -1 & -5 & 0 & 0 \\ 0 & 1 & -1 & 3 & -3 \\ 3 & 2 & -8 & 9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -4 \\ -7 \end{bmatrix}.$$

Use this fact and your work from 2a) to write down the general solution to the inhomogeneous matrix equation $A\underline{x} = \underline{b}$, where $\underline{b} = [1, -2, -4, -7]^T$ is the vector on the right side of the matrix equation above. Explain your reasoning.

(5 points)

3) Find eigenvalues and eigenspace bases for the matrix below. Be careful to check your work, because this matrix shows up in later problems. Hint: the eigenvalues should be negative integers.

$$A := \begin{bmatrix} -4 & 4 \\ 2 & -6 \end{bmatrix}.$$

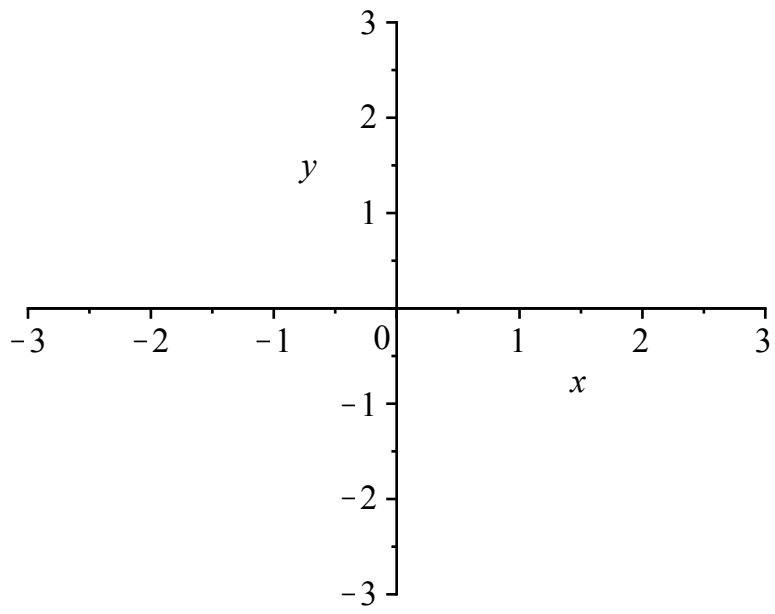
(15 points)

4) Consider the first order system of differential equations, that uses the matrix from the previous exercise:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

4a) Use your work from 3 to write down the general solution to this system of differential equations (4 points)

4b) Classify the equilibrium point at the origin. (Recall, the names we give these equilibrium points are nodal sink, nodal source, saddle point, spiral sink, spiral source, and stable center.) Then sketch the phase portrait for this linear system, using the eigendata of the matrix and the general solution from a. (6 points)



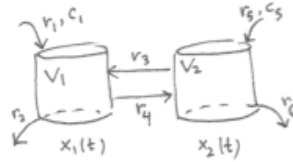
4c) Consider the second order system of differential equations that has the same matrix:

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This system could arise from a two mass, three spring system of the sort we have discussed in class.
What is the general solution in this case?

(5 points)

5) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t), x_2(t)$ respectively. The flow rates (volume per time) are indicated by $r_i, i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5 .



a) What equalities between the flow rates r_1, r_2, \dots, r_6 are necessary and sufficient to guarantee that the volumes V_1, V_2 remain constant? (4 points)

b) Assuming the equalities in a hold, what first order system of differential equations governs the rates of change for $x_1(t), x_2(t)$? (6 points)

c) Suppose $r_1 = 0, r_2 = 200, r_3 = 400, r_4 = 200, r_5 = 400, r_6 = 200 \frac{gal}{hour}$; $c_1 = 0, c_5 = 0.15 \frac{lb}{gal}$; $V_1 = V_2 = 100 gal$. Verify that the constant volumes are consistent with the rate balancing required in a. Then show that the general system in b reduces to the following system of DEs for the given parameter values:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \end{bmatrix}.$$

(4 points)

d) Find the general solution to the system of differential equations in part c. Hint: there is a particular solution that is a constant vector. Note too that the matrix in this system is the same as in problems 3,4. For your convenience, the system is repeated below:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \end{bmatrix}.$$

(6 points)

e) Solve the IVP in part c,d assuming there is initially no solute in either tank, i.e. $x_1(0) = 0, x_2(0) = 0$.

(5 points)

6) Consider the initial value problem below for an undamped mass-spring configuration subject to a constant force for the first one second of motion, but unforced thereafter:

$$\begin{aligned}x''(t) + x(t) &= f(t) \\x(0) &= 0 \\x'(0) &= 0,\end{aligned}$$

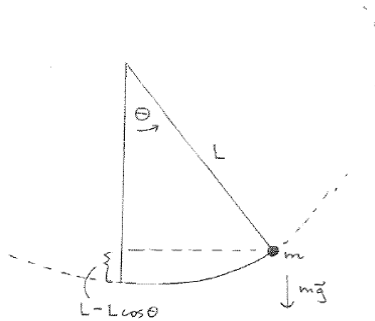
with

$$f(t) = \begin{cases} 10, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$$

Use Laplace transform techniques to solve this IVP.

(10 points)

7) We have studied and returned to the rigid rod pendulum several times in this course. This is the freely-rotating configuration indicated in the diagram below, in which the rod is assumed to be massless of length L , with a mass m at the end, and with angle displacement $\theta(t)$ measured counterclockwise from the vertical.



Using conservation of energy in the no-drag case, we have derived the autonomous second order differential equation that describes the angle $\theta(t)$, arriving at

$$\theta''(t) + \frac{g}{L} \sin(\theta(t)) = 0.$$

7a) Derive the differential equation above, using conservation of energy. Hint: Express the total energy in terms of $\theta(t)$ and $\theta'(t)$ and set its time derivative equal to zero since the total energy function is constant once the pendulum is set in motion.

(5 points)

7b) The second order DE in this problem,

$$\theta''(t) + \frac{g}{L} \sin(\theta(t)) = 0$$

is equivalent to the autonomous first order system of two differential equations

$$x'(t) = y$$

$$y'(t) = -\frac{g}{L} \sin(x) .$$

Explain this equivalence.

(4 points)

7c) Find all equilibrium (i.e. constant) solutions to the first order system of DE's above. Explain how these equilibrium solutions are related to the constant solutions and corresponding configurations of the second order differential equation for rigid-rod pendulum.

(6 points)

7d) For concreteness, suppose that L has been chosen to that $\frac{g}{L} = 4$, i.e. so that the system is given by

$$\begin{aligned}x'(t) &= y \\ y'(t) &= -4 \sin(x).\end{aligned}$$

Use linearization and the Jacobian matrix to classify the equilibrium solutions to the first order system above. Indicate what you can (and cannot) deduce about the stability of these equilibrium solutions based only on the linearization and the eigenvalues. (Hint: recall the names we give to the various possible equilibrium points: nodal source, nodal sink, saddle point, spiral source, spiral sink, stable center.)
(10 points)

7e) Use the total energy function from part a, re-expressed in terms of x, y to explain why the equilibrium solutions that were "borderline" in your linearization analysis are actually stable (but not asymptotically stable). Also, explain what this means in terms of the original pendulum problem.
(5 points)

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$\cos kt$	$\frac{s}{s^2 + k^2}$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\sin kt$	$\frac{k}{s^2 + k^2}$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2 - k^2}$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2 - k^2}$
$e^{at} f(t)$	$F(s-a)$	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2 + k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at} \sin kt$	$\frac{k}{(s-a)^2 + k^2}$
$\int_0^t f(\tau)g(t-\tau) d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt - kt \cos kt)$	$\frac{1}{(s^2 + k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k} \sin kt$	$\frac{s}{(s^2 + k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt \cos kt)$	$\frac{s^2}{(s^2 + k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma) d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)\lfloor t/a \rfloor$ (square wave)	$\frac{1}{s} \tanh \frac{as}{2}$
t	$\frac{1}{s^2}$	$\left\lfloor \frac{t}{a} \right\rfloor$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		