Name ~~S~~OLUTIONS

Student I.D. ______________________

Math 2250-4
Exam 2
November 15 2013

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. There is a Laplace transform table at the end of this test. **Good Luck!**

	Score	POSSIBLE
1		20
2		20
3		20
4		15
5		10
6		15
TOTAL		100

1) Here is a matrix and its reduced row echelon form:

$$A := \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 3 & 1 & 0 \\ -3 & 1 & -1 & -2 & 0 & 0 \\ 1 & -2 & -3 & 1 & -3 & 0 \\ 0 & 1 & 2 & -4 & -2 & 0 \end{array}\right] \quad \text{reduced row echelon form of A:} \quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1a) Find the solution space of vectors $\underline{x}$ that solve the homogeneous matrix equation $A\underline{x} = \underline{0}$. Write your explicit solution (with free parameters), in linear combination form.

(10 points)

backsolve,

$x_1 = -p$
$x_2 = -2p - 2t$
$x_3 = p$
$x_4 = -t$
$x_5 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -p \\ -2p-2t \\ p \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + p\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$t, p \in \mathbb{R}$.

1b) Exhibit a basis for the solution space you found in part 1a.

(2 points)

$$\begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

1c) Define what it means for a collection of vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ ~~to be~~ are linearly independent.

(2 points)

means that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \Rightarrow c_1 = c_2 = \cdots = c_k = 0$

(i.e. the only linear combination that can equal zero is the one where each linear combo coefficient equals 0.)

1d) Are the vectors $\underline{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 0 \\ -1 \\ -3 \\ 2 \end{bmatrix}$ linearly independent or linearly dependent?

Carefully explain why or why not. Hint: these vectors are the first three columns in the matrix A on page 1. The reduced row echelon form result for A from page 1, and reproduced below, may be helpful:

$$A := \begin{array}{c} \begin{bmatrix} 2 & -1 & 0 & 3 & 1 \\ -3 & 1 & -1 & -2 & 0 \\ 1 & -2 & -3 & 1 & -3 \\ 0 & 1 & 2 & -4 & -2 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & & \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & & \end{array} \end{array}; \quad \text{reduced row echelon form of A:} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & & \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & & \end{array} \end{array}.$$

(6 points)

Since $\vec{w}_3 = \vec{w}_1 + 2\vec{w}_2$ it is also true that

$$\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$$

thus $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are <u>not</u> linearly ind., i.e. they are <u>dependent</u>

2a) Consider the linear homogeneous differential equation for $y(x)$:

$$y'''(x) + 4y''(x) + 4y'(x) = 0.$$

Find a basis of solution functions for this homogeneous differential equation. Hint: use the characteristic polynomial.

(10 points)

$$p(r) = r^3 + 4r^2 + 4r$$
$$= r(r^2 + 4r + 4)$$
$$= r(r+2)^2$$

So $e^{0x} = 1, e^{-2x}, xe^{-2x}$ is a basis for the 3-d soltn space.

2b) Use your work from 2a and the method of undetermined coefficients to find the general solution $y(x)$ to

$$y'''(x) + 4y''(x) + 4y'(x) = 8x.$$

$Y = Y_p + y_H$

(10 points)

Since $(r-0)^1$ is a factor of $p(r)$

we try $y_p(x) = x^1(Ax+B)$
$= Ax^2 + Bx.$

$+0(y_p = Ax^2 + Bx)$
$+4(y_p' = 2Ax + B)$
$+4(y_p'' = 2A)$
$1(y_p''' = 0)$

$$\boxed{\begin{aligned} y &= y_p + y_H \\ &= x^2 - 2x + c_1 + c_2 e^{-2x} \\ &\quad + c_3 x e^{-2x} \end{aligned}}$$

$L(y_p) = 8Ax + 4B + 8A$
$= x(8A)$ want $x(8)$
$+ 1(4B + 8A)$ $+1(0)$

$\Rightarrow 8A = 8 \Rightarrow A = 1$
$\Rightarrow 4B + 8 = 0 \Rightarrow 4B = -8$
$\Rightarrow B = -2$ so $\boxed{y_p(x) = x^2 - 2x}$

$y = y_p + y_H$

3a) Use Chapter 5 techniques to solve the initial value problem for this forced damped harmonic oscillator differential equation:

$$x''(t) + 2\,x'(t) + 5\,x(t) = 10\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 1$$

You may use the fact that this differential ~~equation has a particular~~ solution

$$x_p(t) = 2\cos(t) + \sin(t)\ .$$

(You do not need to check this fact; you can just use it.)

(12 points)

$x = x_p + x_H$

x_H: $p(r) = r^2 + 2r + 5$
$= (r+1)^2 + 4$

roots $(r+1)^2 + 4 = 0$
$(r+1)^2 = -4$
$r + 1 = \pm 2i$
$r = -1 \pm 2i$

$x_H(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$x(t) = 2\cos t + \sin t + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$x(0) = 0 = 2 + c_1 \Rightarrow c_1 = -2$

$x'(0) = 1 = 1 - c_1 + 2c_2$

$\Rightarrow 1 = 1 + 2 + 2c_2$

$\Rightarrow -2 = 2c_2 \Rightarrow c_2 = -1$

$$\boxed{x(t) = 2\cos t + \sin t - 2e^{-t}\cos 2t - e^{-t}\sin 2t}$$

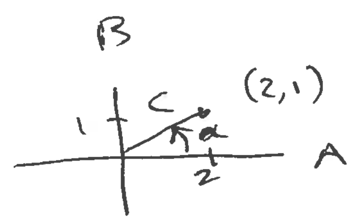
3b) Identify the "steady periodic" and "transient" parts of your solution to 3a. Then express the steady periodic part in amplitude-phase form. (Express the phase angle α as an inverse trig function; you don't need a decimal value.) Hint: You have enough information to do this problem even if you didn't solve the initial value problem in part 3a.

(8 points)

$x_{sp}(t) = 2\cos t + \sin t$

$x_{tr}(t) = -2e^{-t}\cos 2t + e^{-t}\sin 2t$

$= C\cos(t - \alpha)$



$C = \sqrt{A^2 + B^2} = \sqrt{4+1} = \sqrt{5}$

$\alpha = \arctan(B/A) = \arctan(.5)$

4) Re-solve the IVP from problem 3,

$$x''(t) + 2x'(t) + 5x(t) = 10\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 1$$

using Laplace transform techniques (and the table at the end of the test). Use the particular solution $x_p(t) = 2\cos(t) + \sin(t)$ given in problem 3 to deduce two of the four partial fraction coefficients for $X(s)$. This will save you time in figuring out the other two coefficients (which should also be consistent with the rest of your solution to 3, but which we would like you to justify algebraically with partial fraction techniques).

(15 points)

$$s^2 X(s) - \cancel{s\cdot 0} - 1 + 2(sX(s) - \cancel{0}) + 5X(s) = 10\frac{s}{s^2+1}$$

$$X(s)(s^2+2s+5) = 10\frac{s}{s^2+1} + 1.$$

$$X(s) = \frac{10s}{(s^2+1)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$X(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

(from #3, $x_p(t) = 2\cos t + \sin t$ so $A=2,\ B=1.$)

$$\frac{10s}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$$

$$10s = (As+B)(s^2+2s+5) + (Cs+D)(s^2+1)$$
$$10s = (2s+1)(s^2+2s+5) + (Cs+D)(s^2+1)$$

$$0s^3 + 0s^2 + 10s + 1s^0 = s^3(2+C) + s^2(+ s(+ 1(5+D)$$

s^3: $0 = 2 + C \Rightarrow \boxed{C=-2}$

s^0: $0 = 5 + D \Rightarrow \boxed{D=-5}$

$$X(s) = \frac{2s+1}{s^2+1} + \frac{-2s-5}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

$$X(s) = \frac{2s+1}{s^2+1} + \frac{-2s-4}{s^2+2s+5}$$

$$X(s) = \frac{2s+1}{s^2+1} + \frac{-2(s+1)-2}{(s+1)^2+4}$$

$$\boxed{x(t) = 2\cos t + \sin t - 2e^{-t}\cos 2t - e^{-t}\sin 2t}$$

5a) Use Laplace transforms (and the table you've been provided) to solve the forced oscillator initial value problem

$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

(8 points)

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m} \frac{s}{s^2+\omega_0^2}$$

$$X(s)(s^2+\omega_0^2) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2} + s x_0 + v_0$$

$$X(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2} + \frac{s x_0}{s^2+\omega_0^2} + v_0 \frac{1}{s^2+\omega_0^2}$$

table:

$$\boxed{x(t) = \frac{F_0}{m}\frac{t}{2\omega_0}\sin\omega_0 t + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

5b) What word describes the sort of behavior exhibited by solutions to the differential equation in 5a?

(2 points)

(pure) resonance

6) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c \geq 0, F_0 \geq 0$.

6a) Explain what each of the letters m, k, c, F_0, ω represent in this model. Also give their units in the *mks* system.

(10 points)

m = mass kg
k = spring constant N/m
c = damping coeff. Ns/m (or kg/s)
F_0 = forcing amplitude N
ω = angular frequency rad/sec

6b) Describe one of the two forced oscillation phenomena ($F_0 > 0$) that we've studied in this class that is <u>not</u> pure resonance. Use the correct terminology for this phenomenon. Explain what parameter values m, k, ω, c lead to this phenomenon. Indicate which part of the solution formula reflects this phenomenon. (You don't need to exhibit the precise solution formula, just the form it takes.)

(5 points)

<u>practical resonance</u>: $\boxed{x_{sp}(t) = C\cos(\omega t - \alpha)}$

C (and α) depend on ω. If c is small enough, then $C(\omega)$ has a local maximum value near $\omega = \omega_0$, and this is practical resonance. Less precisely, but also acceptable, is to say that this occurs for small damping coeff c, when the amplitude of the steady periodic response is "large" compared to $\frac{F_0}{m}$.

<u>beating</u> $c = 0$, $\omega \approx \omega_0$ but $\omega \neq \omega_0$

part of the solution will be of the form

$\boxed{C(\cos\omega t - \cos\omega_0 t)}$ and for certain t intervals the two terms mostly cancel each others, whereas for complementary ones they have the same sign and add up

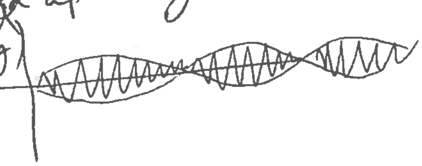
by trig identities this difference of cosines is a product of ~~cosines~~ sines (one with long ~~frequency~~ period) explaining why soln graph looks like 

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		