Name ___Solutions___

Student I.D. _____

Math 2250-4
Exam #1
October 4, 2013

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not a "graphing calculator" i.e. not one which is capable of integration, taking derivatives, or matrix algebra. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

Score

POSSIBLE

1_____10

2_____25

3_____20

4_____15

5_____20

6_____10

TOTAL_____100

1) Concepts:

1a) What is a <u>linear</u> differential equation?

    one that can be written in the form (for $y = y(x)$)   (2 points)

$$y' + P(x)y = Q(x)$$

1b) What is an <u>autonomous</u> differential equation?

    of the form   (2 points)

$$y' = f(y)$$

1c) What is an <u>equilibrium solution</u> to an autonomous differential equation?

    (2 points)

    a constant solution

$$(\text{so } y(x) \equiv c \text{ where } f(c) = 0)$$

1d) For a square matrix $A_{n \times n}$ what condition on the reduced row echelon form of A is equivalent to A having an inverse matrix A^{-1}?

    (2 points)

    A^{-1} exists if and only if

        reduced row echelon form of A is I

1e) For a general rectangular matrix $A_{m \times n}$ what is the condition on the reduced row echelon form of A that guarantees that each matrix equation $A\underline{x} = \underline{b}$ will be consistent (i.e. have at least one solutions $\underline{x}$), for every possible choice of the vector $\underline{b}$?

    (2 points)

    If rref(A) has no zero rows

      (i.e. each row has a leading 1)

    then every system $A\vec{x} = \vec{b}$ is consistent.

2) Consider a boat which starts at rest at time $t = 0$ sec, is accelerated in a straight path by an engine that provides a constant 1500 N of force, and which is also subject to drag forces of 60 N for each $\frac{m}{s}$ of velocity. The boat (containing the pilot) has mass 300 kg.

2a) Use your modeling ability to show that the boat velocity $v(t)$ (in meters per second) satisfies the initial value problem

$$v'(t) = 5 - .2v$$
$$v(0) = 0$$

(5 points)

Newton, 2nd: $mv' = $ net forces

$$300\, v' = 1500 - 60v$$
$$\Rightarrow v' = \frac{1500}{300} - \frac{60}{300}v$$
$$\boxed{v' = 5 - \frac{1}{5}v}$$

because boat starts at rest, $\boxed{v(0) = 0}$

2b) Solve the initial value problem in **2a.** Use the algorithm for separable differential equations.

(15 points)

$$v' = -.2v + 5$$
$$v' = -.2\left(v + \frac{5}{-.2}\right)$$
$$\frac{dv}{dt} = -.2\,(v - 25)$$
$$\frac{dv}{v - 25} = -.2\,dt$$

$$\int : \quad \ln|v - 25| = -.2t + C_1$$

exp: $\quad |v - 25| = e^{C_1}e^{-.2t}$
$$v - 25 = Ce^{-.2t}$$
$$v = 25 + Ce^{-.2t}$$

$v(0) = 0 \Rightarrow 0 = 25 + C \Rightarrow C = -25$

$$\boxed{v(t) = 25 - 25e^{-.2t}}$$
$$= 25\cdot(1 - e^{-.2t})$$

2c) How far does the boat travel in the first ten seconds?

(5 points)

$$x(10) - x(0) = \int_0^{10} x'(t)\,dt$$
$$= \int_0^{10} 25\,(1 - e^{-.2t})\,dt$$
$$= 25\left(t - \frac{e^{-.2t}}{-.2}\right)\Big]_0^{10}$$
$$= 25\left(t + 5e^{-.2t}\right)\Big]_0^{10} = 25\left(10 + 5e^{-2} - (5)\right)$$
$$= 25(5 + 5e^{-2}) = 125(1 + e^{-2}) \text{ meters}$$

3) Consider the following input-output model: A brine tank initially contains 1600 gallons of brine – with an initial concentration is 0.1 *lb* of salt for each gallon of water. At time $t = 0$ water begins to flow into and out of the tank at a rate of 80 gallons per minute, keeping the water volume constant. The salt concentration of the incoming water varies with time, with concentration $2 e^{-0.05 t}$ pounds per gallon, and the brine solution in the tank remains well mixed so that the concentration of brine leaving the tank is always the average concentration in the tank. (There is space below for you to sketch the configuration, as it may help you with the rest of the problem.)

$$80 = r_i$$
$$2e^{-.05t} = c_i$$

$V = 1600$

$v_o = 80$
$c_o = \frac{x(t)}{V}$

$x(t)$

$$x(0) = (.1)(1600) = 160$$
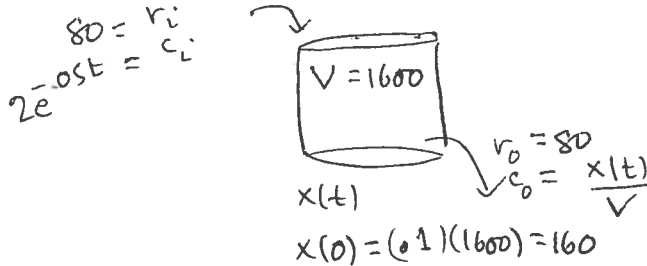
3a) Use the description above to show the salt amount $x(t)$ pounds at time t minutes solves the differential equation

$$x'(t) = 160 e^{-0.05 t} - .05 x(t) .$$

(6 points)

$$x'(t) = r_i c_i - r_o c_o$$
$$= 80 \left(2 e^{-.05t} \right) - 80 \frac{x(t)}{1600}$$

$$\boxed{\begin{array}{l} x'(t) = 160 e^{-.05t} - \frac{1}{20} x(t) \\ = 160 e^{-.05t} - .05 x(t) \end{array}}$$

$$\frac{80}{1600} = \frac{10}{200} = \frac{1}{20}$$

3b) Solve the initial value problem for the salt amount $x(t)$ in this problem.

(14 points)

$$x(0) = (.1) \, 1600 = 160 \text{ lb}.$$
$$\frac{lb}{g} \, g$$

$$x' + .05 x = 160 e^{-.05t}$$

$$e^{.05t} \left(x' + .05 x \right) = e^{.05t} \, 160 \, e^{-.05t} = 160$$

$$e^{.05t} x = \int 160 \, dt = 160 t + C$$

$$\div e^{.05t}$$

$$x = 160 t \, e^{-.05t} + C e^{-.05t}$$

$$x(0) = 160 \Rightarrow 160 = 0 + C \Rightarrow C = 160$$

$$\boxed{\begin{array}{l} x(t) = 160 t \, e^{-.05t} + 160 \, e^{-.05t} \\ = 160 \, e^{-.05t} (t+1) \end{array}}$$
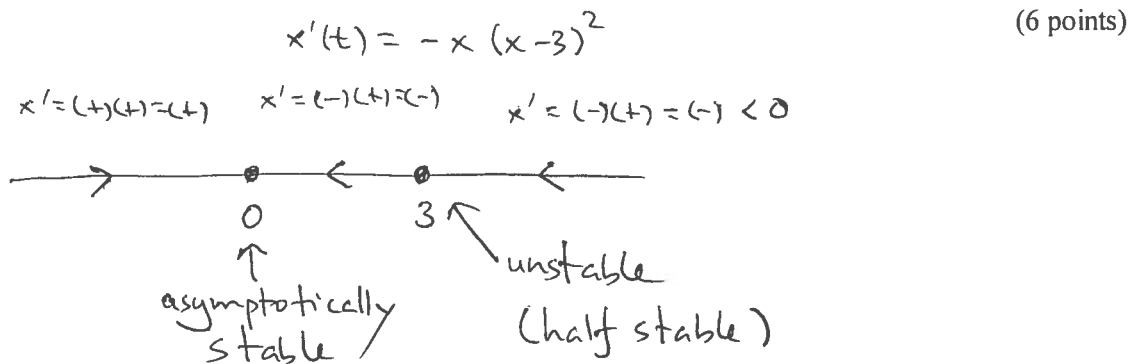
4) Consider the differential equation for $x(t)$:
$$x'(t) = -x^3 + 6x^2 - 9x.$$

4a) Find the equilibrium solutions.

(4 points)

$$x'(t) = -x^3 + 6x^2 - 9x$$
$$= -x[x^2 - 6x + 9]$$
$$x'(t) = -x(x-3)^2$$

so equil. soltns are $x = 0, 3$

4b) Construct the phase diagram for this differential equation, and use it to determine the stability of the equilibrium solutions.

(6 points)

$$x'(t) = -x(x-3)^2$$

$x' = (+)(+) = (+)$ $x' = (-)(+) = (-)$ $x' = (-)(+) = (-) < 0$

asymptotically stable

unstable
(half stable)

4c) Let $x(t)$ be the solution to the initial value problem for this differential equation, with $x(0) = 4$. Without solving for $x(t)$ deduce the value of $\lim_{t \to \infty} x(t)$. Explain.

(5 points)

if $x(0) = 4$ then $\lim_{t \to \infty} x(t) = 3$.

why: • $x(t)$ is decreasing.
• it becomes less than any number greater than 3 in a finite amount of time
  (it leaves each interval $[3+\delta, 4]$ in finite time for each $\delta > 0$).
• $x(t)$ can not reach 3 in finite time t_1 because then $x(t) \equiv 3$ would have to hold because of uniqueness theorem for IVP's.

5) Consider the homogeneous linear system ↙ A

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

5a) Compute the determinant of the matrix in this system. (Be careful to check your work!)

(4 points)

e.g. top row

$$|A| = 2 \begin{vmatrix} 2 & -1 \\ 0 & 6 \end{vmatrix} - 3 \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix} = 2 \cdot 12 - 3 \cdot 8 = 24 - 24 = \boxed{0}$$

5b) What does the value of the determinant in 5a tell you about the possibilities, for the number of solutions to this homogeneous system? Explain.

(4 points)

● because RHS $= \vec{0}$ (homogeneous system), it's a consistent system → in fact $\vec{x} = \vec{0}$ is one solution

● set $|A| = 0$, rref$(A) \neq I$ so there are free parameters in sol'n ⟹ 【infinitely many solutions】

5c) Find the solution set to the linear system above, by computing the reduced row echelon form of the augmented matrix and backsolving.

(12 points)

$$\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 2 & 0 & 6 & 0 \end{array}$$

R_1
R_2

$$\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & 0 \\ 2 & 0 & 6 & 0 \end{array}$$

$-2R_1 + R_2$
$-2R_1 + R_3$

$$\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 8 & 0 \end{array}$$

$-R_2$

$$\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -4 & 8 & 0 \end{array}$$

$4R_2 + R_3$

$$\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$-2R_2 + R_1$

$$\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

backsolve!
$$\boxed{\begin{array}{l} x = -3t \\ y = 2t \\ z = t \in \mathbb{R} \end{array}}$$

$$\left(\text{in linear combination form} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}. \right)$$

6a) Find the inverse matrix to

$$A = \begin{bmatrix} -2 & 3 \\ -3 & 5 \end{bmatrix}.$$

You may use either method we learned.

(5 points)

$|A| = -10 + 9 = -1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ -3 & 2 \end{bmatrix}$$

check:

$$\begin{bmatrix} -2 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ✓

6b) Use your work from 6a to solve the matrix equation below for the unknown matrix X:

$$X \begin{bmatrix} -2 & 3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

$$XA = B \Longleftrightarrow XAA^{-1} = BA^{-1}$$
$$\Longleftrightarrow XI = BA^{-1}$$
$$\Longleftrightarrow X = BA^{-1}$$

(5 points)

$$X = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -3 & 5 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -29 & 18 \\ -13 & 8 \end{bmatrix}.$$

check:

$$\begin{bmatrix} -29 & 18 \\ -13 & 8 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 58-54 & -87+90 \\ 26-24 & -39+40 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$ ✓