Math 2250-4
Fri Dec 6
7.4 forced and unforced mass-spring systems

Warm up exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1b)

$$\begin{bmatrix} x_1{}'' \\ x_2{}'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

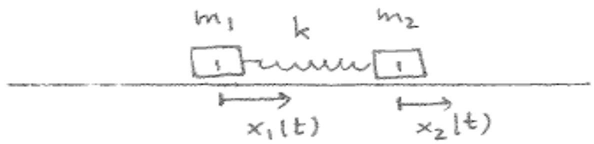
eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

for the eigenvalue $\lambda = -5$, $\underline{\boldsymbol{v}} = [-2, 1]^T$ is an eigenvector; for the eigenvalue $\lambda = -1$, $\underline{\boldsymbol{v}} = [2, 1]^T$ is an eigenvector

- Finish Wednesday's discussion of forced undamped oscillations. Then, continue with today's notes.

Exercise 2) Consider a train with two cars connected by a spring:



2a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}(x_2 - x_1)$$

$$x_2'' = -\frac{k}{m_2}(x_2 - x_1)$$

2b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ verify that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

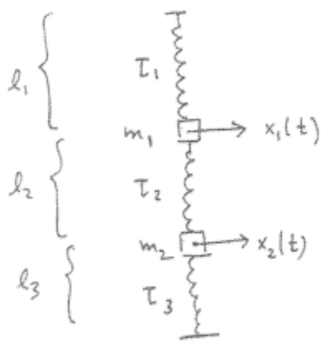
rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems due Monday.

> $\mathit{Eigenvectors}\left(\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\right);$

$$\begin{bmatrix} 0 \\ -\frac{k(m_1+m_2)}{m_2 m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \quad \textbf{(1)}$$

There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions ⊥ to the mass-spring configuration)



T_1, T_2, T_3 are the tensions (pulling forces) of the stretched springs

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1+K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 \quad -(K_2+K_3)x_2$$

where $K_1, K_2 K_3$ are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2^{nd} order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.

horiz force from top spring on mass 1

$$= -T_1 \sin\theta_1 = -T_1 \frac{x_1}{\sqrt{\ell_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{\ell_1} = -\frac{T_1}{\ell_1} x_1$$

So $K_1 = \frac{T_1}{\ell_1}$

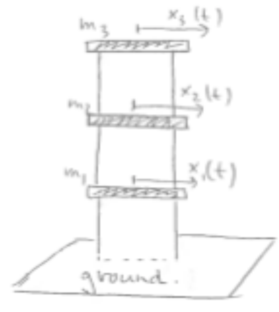
similarly, $K_2 = \frac{T_2}{\ell_2}, K_3 = \frac{T_3}{\ell_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $T \cong k\ell$; $\frac{T}{\ell} \approx k$, so actually almost recover same fundamental modes !!]

- An interesting shake-table demonstration:

http://www.youtube.com/watch?v=M_x2jOKAhZM

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:



For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m, are given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Here is eigendata for the unscaled matrix $\left(\frac{k}{m} = 1\right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$. Symmetric matrices like ours (i.e matrix equals its transpose) are always diagonalizable with real eigenvalues and eigenvectors...and you can choose the eigenvectors to be mutually perpendicular. This is called the "Spectral Theorem for symmetric matrices" and is an important fact in many math and science applications...you can read about it here: http://en.wikipedia.org/wiki/Symmetric_matrix.) If we tell Maple that our matrix is symmetric it will not confuse us with unsimplified numbers and vectors that may look complex rather than real.

```
> with(LinearAlgebra):
> A := Matrix(3, 3, [-2.0, 1, 0, 1, -2, 1, 0, 1, -1]);
      # I used at least one decimal value so Maple would evaluate in floating point
```

$$A := \begin{bmatrix} -2.0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \textbf{(2)}$$

```
> Digits := 5: # 5 digits should be fine, for our decimal approximations.
```

```
> eigendata := Eigenvectors(Matrix(A, shape=symmetric)) : # to take advantage of the
               # spectral theorem
  lambdas := eigendata[1] :  #eigenvalues
  evectors := eigendata[2] :  #corresponding eigenvectors - for fundamental modes
  omegas := map(sqrt, -lambdas);  # natural angular frequencies
  f := x -> 2·evalf(Pi)/x :
  periods := map(f, omegas); #natural periods
  eigenvectors := map(evalf, evectors); #  get digits down to 5
```

$$omegas := \begin{bmatrix} 1.8019 \\ 1.2470 \\ 0.44504 \end{bmatrix}$$

$$periods := \begin{bmatrix} 3.4870 \\ 5.0386 \\ 14.118 \end{bmatrix}$$

$$eigenvectors := \begin{bmatrix} -0.59101 & -0.73698 & 0.32799 \\ 0.73698 & -0.32799 & 0.59101 \\ -0.32799 & 0.59101 & 0.73698 \end{bmatrix} \quad \textbf{(3)}$$

```
> B := Matrix(2, 2, [-2, 1.0, 1, -1]) :
  eigendata := Eigenvectors(Matrix(B, shape=symmetric)) : # to take advantage of the
               # spectral theorem
  lambdas := eigendata[1] :  #eigenvalues
  evectors := eigendata[2] :  #corresponding eigenvectors - for fundamental modes
  omegas := map(sqrt, -lambdas);  # natural angular frequencies
  f := x -> 2·evalf(Pi)/x :
  periods := map(f, omegas); #natural periods
  eigenvectors := map(evalf, evectors); #  get digits down to 5
```

$$omegas := \begin{bmatrix} 1.6180 \\ 0.61804 \end{bmatrix}$$

$$periods := \begin{bmatrix} 3.8834 \\ 10.166 \end{bmatrix}$$

$$eigenvectors := \begin{bmatrix} -0.85065 & -0.52573 \\ 0.52573 & -0.85065 \end{bmatrix} \quad \textbf{(4)}$$

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>
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Exercise 3) Interpret the data above, in terms of the natural modes for the shaking building . In the youtube video the first mode to appear is the slow and dangerous "sloshing mode", where all three floors oscillate in phase, with amplitude ratios 33 : 59 : 74 from the first to the third floor. What's the second mode that gets excited? The third mode? (They don't show the third mode in the video.)

Remark) All of the ideas we've discussed in section 7.4 also apply to molecular vibrations. The eigendata in these cases is related to the "spectrum" of light frequencies that correspond to the natural fundamental modes for molecular vibrations.