Math 2250-4
Tues Dec 3

Finish 7.3 Applications of first order systems of differential equations

· Begin by finishing the **Glucose-insulin model** in Monday's notes. The IVP and some calculation details that we completed on Monday are below for quick reference, but the rest of this discussion will use yesterday's notes.

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

· Characteristic polynomial $|A - \lambda I| = (\lambda + .1)^2 + .04$ has roots $\lambda = -.1 \pm .2 \, i$.
· For the eigenvalue $\lambda = -.1 + .2 \, i$ we wish to solve the eigenvector system $(A - \lambda I)\underline{v} = \underline{0}$:

$$\left[\begin{array}{cc|c} -.2 \, i & -0.4 & 0 \\ 0.1 & -.2 \, i & 0 \end{array} \right]$$

$10 \, R_2 \rightarrow R_1, \ -5 \, R_1 \rightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & -2 \, i & 0 \\ i & 2 & 0 \end{array} \right]$$

Although it doesn't look like it, the second row is a multiple of the first row, as it must be:
$-i R_1 + R_2 \rightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & -2 \, i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

So we may choose the eigenvector $\underline{v} = [2 \, i, \, 1]^T$. This yields a complex function solution

$$\underline{z}(t) = e^{\lambda t}\underline{v} = e^{(-.1 + 2 \, i)t} \begin{bmatrix} 2 \, i \\ 1 \end{bmatrix}.$$

If we rewrite $\underline{z}(t) = \underline{x}(t) + i \underline{y}(t)$ with $\underline{x}(t), \underline{y}(t)$ real vector functions, then each of $\underline{x}(t), \underline{y}(t)$ will be real solutions to the system of differential equations, and will in fact be a basis. (See general discussion at end of Monday's notes.) At the end of class we did this decomposition of $\underline{z}(t)$ into real and imaginary parts:

$$e^{(-.1 + 2 \, i)t} \begin{bmatrix} 2 \, i \\ 1 \end{bmatrix} = e^{-.1 \, t}(\cos(.2 \, t) + i \sin(.2 \, t)) \begin{bmatrix} 2 \, i \\ 1 \end{bmatrix}$$

$$= e^{-.1 \, t} \begin{bmatrix} (\cos(.2 \, t) + i \sin(.2 \, t))2 \, i \\ (\cos(.2 \, t) + i \sin(.2 \, t))1 \end{bmatrix} = e^{-.1 \, t} \begin{bmatrix} -2 \sin(.2 \, t) \\ \cos(.2 \, t) \end{bmatrix} + i \, e^{-.1 \, t} \begin{bmatrix} 2 \cos(.2 \, t) \\ \cos(.2 \, t) \end{bmatrix}.$$

This gives real solutions

$$\underline{x}(t) = e^{-.1 \, t} \begin{bmatrix} -2 \sin(.2 \, t) \\ \cos(.2 \, t) \end{bmatrix}, \underline{y}(t) = e^{-.1 \, t} \begin{bmatrix} 2 \cos(.2 \, t) \\ \sin(.2 \, t) \end{bmatrix}$$

and general homogeneous solution using real functions:

$$\underline{x}_H(t) = c_1\underline{x}(t) + c_2\underline{y}(t) = c_1 e^{-.1 \, t} \begin{bmatrix} -2 \sin(.2 \, t) \\ \cos(.2 \, t) \end{bmatrix} + c_2 e^{-.1 \, t} \begin{bmatrix} 2 \cos(.2 \, t) \\ \sin(.2 \, t) \end{bmatrix}.$$

• The Glucose-insulin example is linearized, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and a recent graduate student Chris Remien in the Math Department, and collaborating with the University Hospital recently modeled liver poisoning by acetominophen (brand name Tyleonol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Here's a link to their published paper. For fun, I copied and pasted the non-linear system of first order differential equations from a preprint of their paper, below:
http://onlinelibrary.wiley.com/doi/10.1002/hep.25656/full
http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf

APAP
$$\frac{dA}{dt} = -\frac{\alpha}{H_{max}}AH - \delta_a A$$

NAPQI
$$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}}A - \gamma NG$$

GSH
$$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$$

Functional Hepatocytes
$$\frac{dH}{dt} = rH\left(1 - \frac{H+Z}{H_{max}}\right) - \eta NH$$

Damaged Hepatocytes
$$\frac{dZ}{dt} = \eta NH - \delta_z Z$$

AST
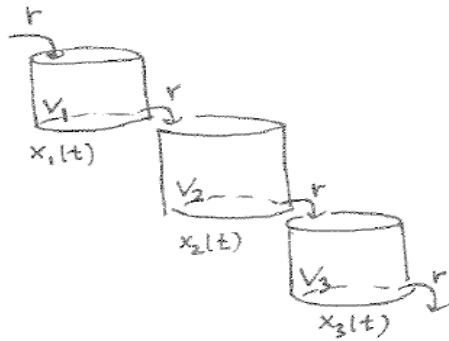$$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}}Z - \delta_s(S - S_{min})$$

ALT
$$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}}Z - \delta_l(L - L_{min})$$

Clotting Factor
$$\frac{dF}{dt} = \beta_f\left(\frac{H}{H_{max}} - F\right)$$

<u>Example 2)</u> consider the three component input-output model below:

<u>Exercise 1a)</u> Derive the first order system for the tank cascade above.

<u>1b)</u> In case the tank volumes (in gallons) are $V_1 = 20$, $V_2 = 40$, $V_3 = 50$, the flow rate $r = 10 \; \dfrac{gal}{min}$, and pure water (with no solute) is flowing into the first (top) tank, show that your system in (a) can be written as

$$\begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \\ x_3{}'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

(This system is actually worked out in the text, page 422-434...but we'll modify the IVP, and then consider a second case as well.)

1c) Here's the eigenvector data for the matrix in **b**. You may want to check or derive parts of it by hand, especially if you're still not expert at finding eigenvalues and eigenvectors. I entered the matrix entries as rational numbers rather than decimals, because otherwise Maple would have given (confusing) floating point answers. Use the eigendata to write down the general solution to the system in **b**.

```
> with(LinearAlgebra) :
> A := Matrix( 3, 3, [ -1/2, 0, 0, 1/2, -1/4, 0, 0, 1/4, -1/5 ] );
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -\dfrac{1}{2} & 0 & 0 \\[6pt] \dfrac{1}{2} & -\dfrac{1}{4} & 0 \\[6pt] 0 & \dfrac{1}{4} & -\dfrac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} -\dfrac{1}{4} \\[4pt] -\dfrac{1}{5} \\[4pt] -\dfrac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dfrac{3}{5} \\[4pt] -\dfrac{1}{5} & 0 & -\dfrac{6}{5} \\[4pt] 1 & 1 & 1 \end{bmatrix}$$

(1)

```
>
```

1d) Solve the IVP for this tank cascade, assuming that there are initially 15 *lb* of salt in the first tank, and no salt in the second and third tanks.

Your answer to d should be

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 5\ e^{-0.5\,t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30\ e^{-0.25\,t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125\ e^{-0.2\,t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
$$

1e) We can plot the amounts of salt in each tank to figure out what's going on. Make sure you understand how the formulas below are related to the vector equation above, and interpret the graphical results.

```
> with(plots) :
> x1 := t→15·exp(−.5·t) :
  plot1 := plot(x1(t), t = 0 ..30, color = black) :
  x2 := t→ −30·exp(−.5·t) + 30·exp(−.25·t) :
  plot2 := plot(x2(t), t = 0 ..30, color = brown) :
  x3 := t→25·exp(−.5·t) − 150·exp(−.25·t) + 125·exp(−.2·t) :
  plot3 := plot(x3(t), t = 0 ..30, color = green) :
  display({plot1, plot2, plot3}, title = `pollutant flushing in tank cascade`);
```

pollutant flushing in tank cascade
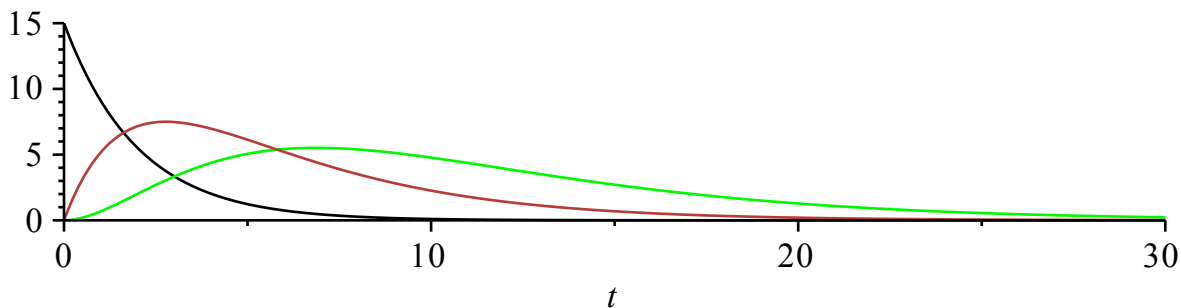
Exercise 2) Use the same tank cascade. Only now, assume that there is initially 13 *lb* salt in the first tank, none in the others, and that when the water starts flowing the input pipe contains salty water, with concentration 0.5 $\dfrac{lb}{gal}$.

2a) Explain why this yields an IVP for an inhomogeneous system of linear DE's, namely

$$
\begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \\ x_3{}'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.
$$

$$
\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 0 \end{bmatrix}.
$$

<u>2b)</u> Use a vector analog of "undetermined coefficients" to guess that there might be a particular solution that is a constant vector, i.e.

$$\underline{x}_P(t) = \underline{c}.$$

Plug this guess into the inhomgeneous system to deduce $\underline{c}$. How could your irritating younger sibling have told you this particular solution, without knowing anything at all about linear algebra or differential equations?

<u>2c)</u> Use $\underline{x}(t) = \underline{x}_P(t) + \underline{x}_H(t)$ to solve the IVP. Compare your solution to the plots below.

```
> x1 := t→10 + 3·exp(-.5·t) :
  plot1 := plot(x1(t), t = 0 ..50, color = black) :
  x2 := t→20 − 6·exp(-.5·t) −14·exp(-.25·t) :
  plot2 := plot(x2(t), t = 0 ..50, color = brown) :
  x3 := t→25 + 5·exp(-.5·t) + 70·exp(-.25·t) − 100·exp(-.2·t) :
  plot3 := plot(x3(t), t = 0 ..50, color = green) :
  display({plot1, plot2, plot3}, title = `pollutant level stabilizing in tank cascade`);
```


pollutant level stabilizing in tank cascade

```
>
```