1 First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Example 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3+7x^2-11x+41=0$. By contrast, in solving a differential equation, we are challenged to find the unknown *functions* $y=y(x)$ for which an identity such as $y'(x)=2xy(x)$—that is, the differential equation

$$\frac{dy}{dx}=2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation, if possible.

Example 2 If C is a constant and

$$y(x)=Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx}=C\left(2xe^{x^2}\right)=(2x)\left(Ce^{x^2}\right)=2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx}=2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant C. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1). ■

Differential Equations and Mathematical Models

The following three examples illustrate the process of translating scientific laws and principles into differential equations. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

Example 3 Newton's law of cooling may be stated in this way: The *time rate of change* (the rate of change with respect to time t) of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium (Fig. 1.1.1). That is,

$$\frac{dT}{dt}=-k(T-A), \tag{3}$$

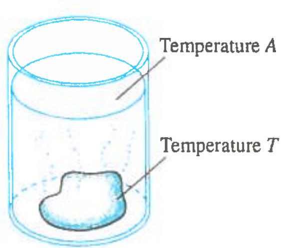

FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

where k is a positive constant. Observe that if $T>A$, then $dT/dt<0$, so the temperature is a decreasing function of t and the body is cooling. But if $T<A$, then $dT/dt>0$, so that T is increasing.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body. ■

Example 4 Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank (Fig. 1.1.2) is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt}=-k\sqrt{y}, \tag{4}$$

where k is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$, so $dV/dt = A \cdot (dy/dt)$. In this case Eq. (4) takes the form

$$\frac{dy}{dt} = -h\sqrt{y}, \tag{5}$$

where $h = k/A$ is a constant. ■

Example 5 The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{6}$$

where k is the constant of proportionality. ■

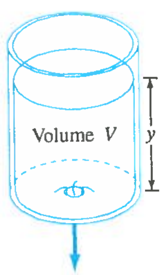

FIGURE 1.1.2. Torricelli's law of draining, Eq. (4), describes the draining of a water tank.

Let us discuss Example 5 further. Note first that each function of the form

$$P(t) = Ce^{kt} \tag{7}$$

is a solution of the differential equation

$$\frac{dP}{dt} = kP$$

in (6). We verify this assertion as follows:

$$P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP(t)$$

for all real numbers t. Because substitution of each function of the form given in (7) into Eq. (6) produces an identity, all such functions are solutions of Eq. (6).

Thus, even if the value of the constant k is known, the differential equation $dP/dt = kP$ has *infinitely many* different solutions of the form $P(t) = Ce^{kt}$, one for each choice of the "arbitrary" constant C. This is typical of differential equations. It is also fortunate, because it may allow us to use additional information to select from among all these solutions a particular one that fits the situation under study.

Example 6 Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at time $t = 0$ (hours, h) was 1000, and that the population doubled after 1 h. This additional information about $P(t)$ yields the following equations:

$$1000 = P(0) = Ce^0 = C,$$
$$2000 = P(1) = Ce^k.$$

It follows that $C = 1000$ and that $e^k = 2$, so $k = \ln 2 \approx 0.693147$. With this value of k the differential equation in (6) is

$$\frac{dP}{dt} = (\ln 2)P \approx (0.693147)P.$$

Substitution of $k = \ln 2$ and $C = 1000$ in Eq. (7) yields the particular solution

$$P(t) = 1000e^{(\ln 2)t} = 1000(e^{\ln 2})^t = 1000 \cdot 2^t \qquad (\text{because } e^{\ln 2} = 2)$$

that satisfies the given conditions. We can use this particular solution to predict future populations of the bacteria colony. For instance, the predicted number of bacteria in the population after one and a half hours (when $t = 1.5$) is

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828.$$ ■

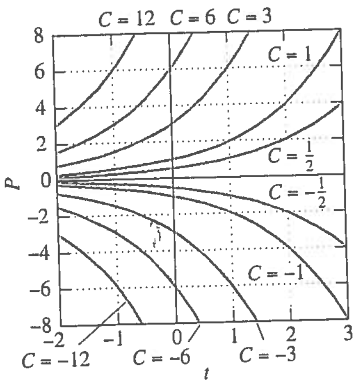

FIGURE 1.1.3. Graphs of $P(t) = Ce^{kt}$ with $k = \ln 2$.

The condition $P(0) = 1000$ in Example 6 is called an **initial condition** because we frequently write differential equations for which $t = 0$ is the "starting time." Figure 1.1.3 shows several different graphs of the form $P(t) = Ce^{kt}$ with $k = \ln 2$. The graphs of all the infinitely many solutions of $dP/dt = kP$ in fact fill the entire two-dimensional plane, and no two intersect. Moreover, the selection of any one point P_0 on the P-axis amounts to a determination of $P(0)$. Because exactly one solution passes through each such point, we see in this case that an initial condition $P(0) = P_0$ determines a unique solution agreeing with the given data.

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

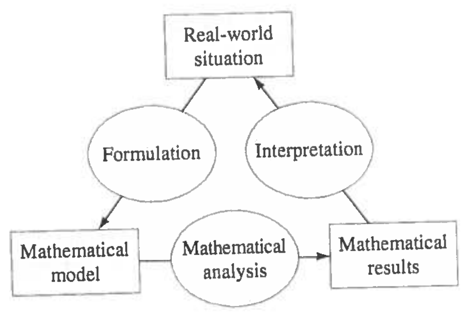

FIGURE 1.1.4. The process of mathematical modeling.

In the population example, the real-world problem is that of determining the population at some future time. A **mathematical model** consists of a list of variables (P and t) that describe the given situation, together with one or more equations relating these variables ($dP/dt = kP$, $P(0) = P_0$) that are known or are assumed to hold. The mathematical analysis consists of solving these equations (here, for P as a function of t). Finally, we apply these mathematical results to attempt to answer the original real-world question.

As an example of this process, think of first formulating the mathematical model consisting of the equations $dP/dt = kP$, $P(0) = 1000$, describing the bacteria population of Example 6. Then our mathematical analysis there consisted of solving for the solution function $P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t$ as our mathematical result. For an interpretation in terms of our real-world situation—the actual bacteria population—we substituted $t = 1.5$ to obtain the predicted population of $P(1.5) \approx 2828$ bacteria after 1.5 hours. If, for instance, the bacteria population is growing under ideal conditions of unlimited space and food supply, our prediction may be quite accurate, in which case we conclude that the mathematical model is adequate for studying this particular population.

On the other hand, it may turn out that no solution of the selected differential equation accurately fits the actual population we're studying. For instance, for *no* choice of the constants C and k does the solution $P(t) = Ce^{kt}$ in Eq. (7) accurately describe the actual growth of the human population of the world over the past few centuries. We must conclude that the differential equation $dP/dt = kP$ is inadequate for modeling the world population—which in recent decades has "leveled off" as compared with the steeply climbing graphs in the upper half ($P > 0$) of Fig. 1.1.3. With sufficient insight, we might formulate a new mathematical model including a perhaps more complicated differential equation, one that takes into account such factors as a limited food supply and the effect of increased population on birth and death rates. With the formulation of this new mathematical model, we may attempt to traverse once again the diagram of Fig. 1.1.4 in a counterclockwise manner. If we can solve the new differential equation, we get new solution functions to compare with the real-world population. Indeed, a successful population analysis may require refining the mathematical model still further as it is repeatedly measured against real-world experience.

But in Example 6 we simply ignored any complicating factors that might affect our bacteria population. This made the mathematical analysis quite simple, perhaps unrealistically so. A satisfactory mathematical model is subject to two contradictory requirements: It must be sufficiently detailed to represent the real-world situation with relative accuracy, yet it must be sufficiently simple to make the mathematical analysis practical. If the model is so detailed that it fully represents the physical situation, then the mathematical analysis may be too difficult to carry out. If the model is too simple, the results may be so inaccurate as to be useless. Thus there is an inevitable tradeoff between what is physically realistic and what is mathematically possible. The construction of a model that adequately bridges this gap between realism and feasibility is therefore the most crucial and delicate step in the process. Ways must be found to simplify the model mathematically without sacrificing essential features of the real-world situation.

Mathematical models are discussed throughout this book. The remainder of this introductory section is devoted to simple examples and to standard terminology used in discussing differential equations and their solutions.

Examples and Terminology

Example 7 If C is a constant and $y(x) = 1/(C - x)$, then

$$\frac{dy}{dx} = \frac{1}{(C - x)^2} = y^2$$

if $x \neq C$. Thus

$$y(x) = \frac{1}{C - x} \tag{8}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \tag{9}$$

on any interval of real numbers not containing the point $x = C$. Actually, Eq. (8) defines a *one-parameter family* of solutions of $dy/dx = y^2$, one for each value of the arbitrary constant or "parameter" C. With $C = 1$ we get the particular solution

$$y(x) = \frac{1}{1 - x}$$

that satisfies the initial condition $y(0) = 1$. As indicated in Fig. 1.1.5, this solution is continuous on the interval $(-\infty, 1)$ but has a vertical asymptote at $x = 1$. ■

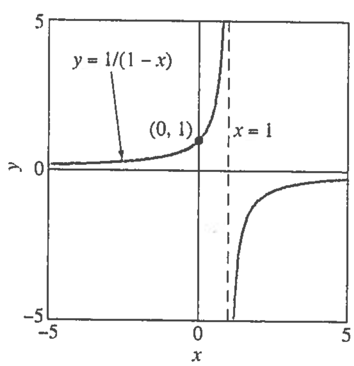

FIGURE 1.1.5. The solution of $y' = y^2$ defined by $y(x) = 1/(1 - x)$.

Example 8 Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln x$ satisfies the differential equation

$$4x^2y'' + y = 0 \tag{10}$$

for all $x > 0$.

Solution First we compute the derivatives

$$y'(x) = -\tfrac{1}{2}x^{-1/2}\ln x \quad \text{and} \quad y''(x) = \tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

$$4x^2y'' + y = 4x^2\left(\tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln x = 0$$

if x is positive, so the differential equation is satisfied for all $x > 0$. ■

The fact that we can write a differential equation is not enough to guarantee that it has a solution. For example, it is clear that the differential equation

$$(y')^2 + y^2 = -1 \tag{11}$$

has *no* (real-valued) solution, because the sum of nonnegative numbers cannot be negative. For a variation on this theme, note that the equation

$$(y')^2 + y^2 = 0 \tag{12}$$

obviously has only the (real-valued) solution $y(x) \equiv 0$. In our previous examples any differential equation having at least one solution indeed had infinitely many.

The **order** of a differential equation is the order of the highest derivative that appears in it. The differential equation of Example 8 is of second order, those in Examples 2 through 7 are first-order equations, and

$$y^{(4)} + x^2y^{(3)} + x^5y = \sin x$$

is a fourth-order equation. The most general form of an ***n*th-order** differential equation with independent variable x and unknown function or dependent variable $y = y(x)$ is

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \tag{13}$$

where F is a specific real-valued function of $n + 2$ variables.

Our use of the word *solution* has been until now somewhat informal. To be precise, we say that the continuous function $u = u(x)$ is a **solution** of the differential equation in (13) **on the interval** I provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F\left(x, u, u', u'', \dots, u^{(n)}\right) = 0$$

for all x in I. For the sake of brevity, we may say that $u = u(x)$ **satisfies** the differential equation in (13) on I.

Remark: Recall from elementary calculus that a differentiable function on an open interval is necessarily continuous there. This is why only a continuous function can qualify as a (differentiable) solution of a differential equation on an interval. ■

Example 7 *Continued*

Figure 1.1.5 shows the two "connected" branches of the graph $y = 1/(1-x)$. The left-hand branch is the graph of a (continuous) solution of the differential equation $y' = y^2$ that is defined on the interval $(-\infty, 1)$. The right-hand branch is the graph of a *different* solution of the differential equation that is defined (and continuous) on the different interval $(1, \infty)$. So the single formula $y(x) = 1/(1-x)$ actually defines two different solutions (with different domains of definition) of the same differential equation $y' = y^2$. ■

Example 9

If A and B are constants and

$$y(x) = A\cos 3x + B\sin 3x, \tag{14}$$

then two successive differentiations yield

$$\begin{aligned} y'(x) &= -3A\sin 3x + 3B\cos 3x, \\ y''(x) &= -9A\cos 3x - 9B\sin 3x = -9y(x) \end{aligned}$$

for all x. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$y'' + 9y = 0 \tag{15}$$

on the whole real number line. Figure 1.1.6 shows the graphs of several such solutions. ■

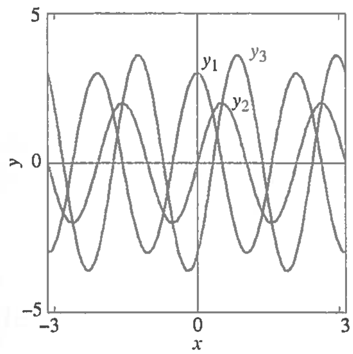

FIGURE 1.1.6. The three solutions $y_1(x) = 3\cos 3x$, $y_2(x) = 2\sin 3x$, and $y_3(x) = -3\cos 3x + 2\sin 3x$ of the differential equation $y'' + 9y = 0$.

Although the differential equations in (11) and (12) are exceptions to the general rule, we will see that an nth-order differential equation ordinarily has an n-parameter family of solutions—one involving n different arbitrary constants or parameters.

In both Eqs. (11) and (12), the appearance of y' as an implicitly defined function causes complications. For this reason, we will ordinarily assume that any differential equation under study can be solved explicitly for the highest derivative that appears; that is, that the equation can be written in the so-called *normal form*

$$y^{(n)} = G\left(x, y, y', y'', \dots, y^{(n-1)}\right), \tag{16}$$

where G is a real-valued function of $n + 1$ variables. In addition, we will always seek only real-valued solutions unless we warn the reader otherwise.

All the differential equations we have mentioned so far are **ordinary** differential equations, meaning that the unknown function (dependent variable) depends on only a *single* independent variable. If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a **partial** differential equation. For example, the temperature $u = u(x, t)$ of a long thin uniform rod at the point x at time t satisfies (under appropriate simple conditions) the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where k is a constant (called the *thermal diffusivity* of the rod). In this book we will be concerned only with *ordinary* differential equations and will refer to them simply as differential equations.

In this chapter we concentrate on *first-order* differential equations of the form

$$\frac{dy}{dx} = f(x, y). \tag{17}$$

We also will sample the wide range of applications of such equations. A typical mathematical model of an applied situation will be an **initial value problem**, consisting of a differential equation of the form in (17) together with an **initial condition** $y(x_0) = y_0$. Note that we call $y(x_0) = y_0$ an initial condition whether or not $x_0 = 0$. To **solve** the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{18}$$

means to find a differentiable function $y = y(x)$ that satisfies both conditions in Eq. (18) on some interval containing x_0.

Example 10 Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Solution We need only find a value of C so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3 - 2x)$. The left-hand branch is the graph on $(-\infty, \frac{3}{2})$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $(\frac{3}{2}, \infty)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

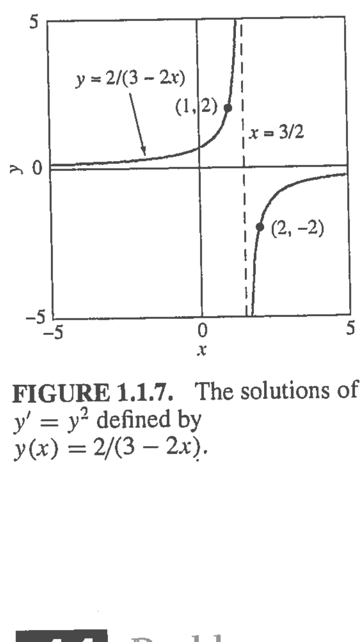

FIGURE 1.1.7. The solutions of $y' = y^2$ defined by $y(x) = 2/(3 - 2x)$.

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1 + x^2}$
10. $x^2y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x\sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^{x} - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2 y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$. Can you *guess* what the graph of such a function g might look like?
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.
34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.
35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.
36. In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. $y'' = 0$
38. $y' = y$
39. $xy' + y = 3x^2$
40. $(y')^2 + y^2 = 1$
41. $y' + y = e^x$
42. $y'' + y = 0$

43. **(a)** If k is a constant, show that a general (one-parameter) solution of the differential equation

$$\frac{dx}{dt} = kx^2$$

is given by $x(t) = 1/(C - kt)$, where C is an arbitrary constant.

(b) Determine by inspection a solution of the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** Continuing Problem 43, assume that k is positive, and then sketch graphs of solutions of $x' = kx^2$ with several typical positive values of $x(0)$.

(b) How would these solutions differ if the constant k were negative?

45. Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

46. Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate of 1 m/s² when $v = 5$ m/s. How long does it take for the velocity of the boat to decrease to 1 m/s? To $\frac{1}{10}$ m/s? When does the boat come to a stop?

47. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. **(a)** Determine a value of C so that $y(10) = 10$. **(b)** Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? **(c)** Figure 1.1.8 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$. Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point (a, b) in the plane, the differential equation $dy/dx = y^2$ has exactly one solution $y(x)$ satisfying the condition $y(a) = b$?

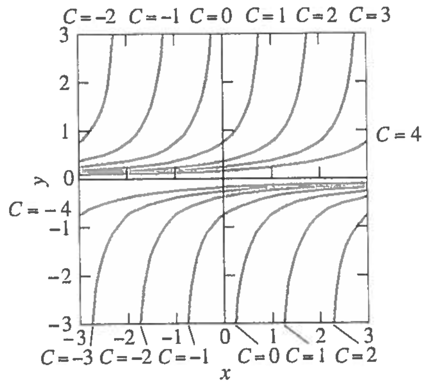

FIGURE 1.1.8. Graphs of solutions of the equation $dy/dx = y^2$.

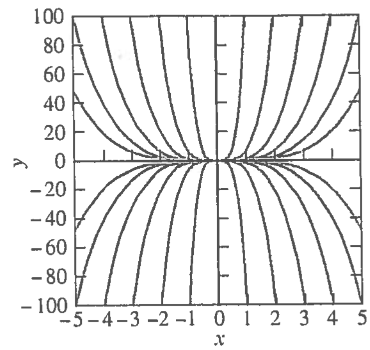

FIGURE 1.1.9. The graph $y = Cx^4$ for various values of C.

48. **(a)** Show that $y(x) = Cx^4$ defines a one-parameter family of differentiable solutions of the differential equation $xy' = 4y$ (Fig. 1.1.9). **(b)** Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geqq 0 \end{cases}$$

defines a differentiable solution of $xy' = 4y$ for all x, but is not of the form $y(x) = Cx^4$. **(c)** Given any two real numbers a and b, explain why—in contrast to the situation in part (c) of Problem 47—there exist infinitely many differentiable solutions of $xy' = 4y$ that all satisfy the condition $y(a) = b$.

1.2 Integrals as General and Particular Solutions

The first-order equation $dy/dx = f(x, y)$ takes an especially simple form if the right-hand-side function f does not actually involve the dependent variable y, so

$$\frac{dy}{dx} = f(x). \tag{1}$$

In this special case we need only integrate both sides of Eq. (1) to obtain

$$y(x) = \int f(x)\,dx + C. \tag{2}$$

This is a **general solution** of Eq. (1), meaning that it involves an arbitrary constant C, and for every choice of C it is a solution of the differential equation in (1). If $G(x)$ is a particular antiderivative of f—that is, if $G'(x) \equiv f(x)$—then

$$y(x) = G(x) + C. \tag{3}$$

The graphs of any two such solutions $y_1(x) = G(x) + C_1$ and $y_2(x) = G(x) + C_2$ on the same interval I are "parallel" in the sense illustrated by Figs. 1.2.1 and 1.2.2. There we see that the constant C is geometrically the vertical distance between the two curves $y(x) = G(x)$ and $y(x) = G(x) + C$.

To satisfy an initial condition $y(x_0) = y_0$, we need only substitute $x = x_0$ and $y = y_0$ into Eq. (3) to obtain $y_0 = G(x_0) + C$, so that $C = y_0 - G(x_0)$. With this choice of C, we obtain the **particular solution** of Eq. (1) satisfying the initial value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.$$

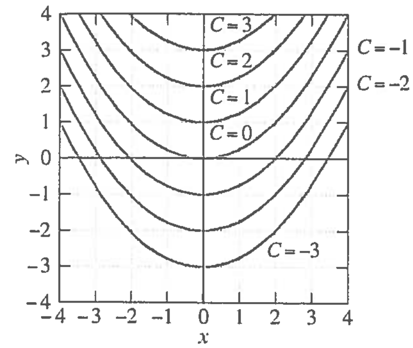

FIGURE 1.2.1. Graphs of $y = \frac{1}{4}x^2 + C$ for various values of C.

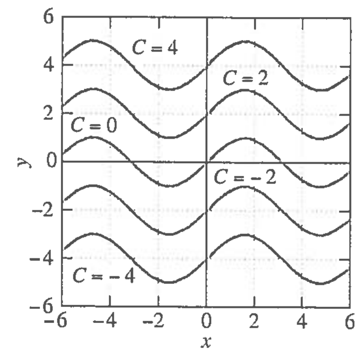

FIGURE 1.2.2. Graphs of $y = \sin x + C$ for various values of C.

We will see that this is the typical pattern for solutions of first-order differential equations. Ordinarily, we will first find a *general solution* involving an arbitrary constant C. We can then attempt to obtain, by appropriate choice of C, a *particular solution* satisfying a given initial condition $y(x_0) = y_0$.

Remark: As the term is used in the previous paragraph, a *general solution* of a first-order differential equation is simply a one-parameter family of solutions. A natural question is whether a given general solution contains *every* particular solution of the differential equation. When this is known to be true, we call it **the** general solution of the differential equation. For example, because any two antiderivatives of the same function $f(x)$ can differ only by a constant, it follows that every solution of Eq. (1) is of the form in (2). Thus Eq. (2) serves to define **the** general solution of (1). ■

Example 1 Solve the initial value problem

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2.$$

Solution Integration of both sides of the differential equation as in Eq. (2) immediately yields the general solution

$$y(x) = \int (2x + 3)\,dx = x^2 + 3x + C.$$

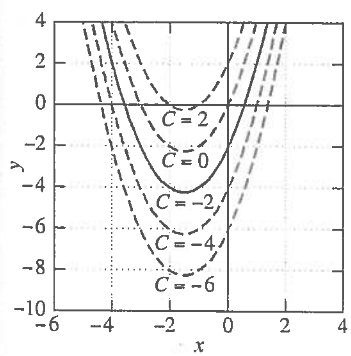

FIGURE 1.2.3. Solution curves for the differential equation in Example 1.

Figure 1.2.3 shows the graph $y = x^2 + 3x + C$ for various values of C. The particular solution we seek corresponds to the curve that passes through the point $(1, 2)$, thereby satisfying the initial condition

$$y(1) = (1)^2 + 3 \cdot (1) + C = 2.$$

It follows that $C = -2$, so the desired particular solution is

$$y(x) = x^2 + 3x - 2. \quad ■$$

Second-order equations. The observation that the special first-order equation $dy/dx = f(x)$ is readily solvable (provided that an antiderivative of f can be found) extends to second-order differential equations of the special form

$$\frac{d^2y}{dx^2} = g(x), \tag{4}$$

in which the function g on the right-hand side involves neither the dependent variable y nor its derivative dy/dx. We simply integrate once to obtain

$$\frac{dy}{dx} = \int y''(x)\,dx = \int g(x)\,dx = G(x) + C_1,$$

where G is an antiderivative of g and C_1 is an arbitrary constant. Then another integration yields

$$y(x) = \int y'(x)\,dx = \int [G(x) + C_1]\,dx = \int G(x)\,dx + C_1x + C_2,$$

where C_2 is a second arbitrary constant. In effect, the second-order differential equation in (4) is one that can be solved by solving successively the *first-order* equations

$$\frac{dv}{dx} = g(x) \quad \text{and} \quad \frac{dy}{dx} = v(x).$$

Velocity and Acceleration

Direct integration is sufficient to allow us to solve a number of important problems concerning the motion of a particle (or *mass point*) in terms of the forces acting on it. The motion of a particle along a straight line (the x-axis) is described by its **position function**

$$x = f(t) \tag{5}$$

giving its x-coordinate at time t. The **velocity** of the particle is defined to be

$$v(t) = f'(t); \quad \text{that is,} \quad v = \frac{dx}{dt}. \tag{6}$$

Its **acceleration** $a(t)$ is $a(t) = v'(t) = x''(t)$; in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{7}$$

Equation (6) is sometimes applied either in the indefinite integral form $x(t) = \int v(t)\,dt$ or in the definite integral form

$$x(t) = x(t_0) + \int_{t_0}^{t} v(s)\,ds,$$

which you should recognize as a statement of the fundamental theorem of calculus (precisely because $dx/dt = v$).

Newton's *second law of motion* says that if a force $F(t)$ acts on the particle and is directed along its line of motion, then

$$ma(t) = F(t); \quad \text{that is,} \quad F = ma, \tag{8}$$

where m is the mass of the particle. If the force F is known, then the equation $x''(t) = F(t)/m$ can be integrated twice to find the position function $x(t)$ in terms of two constants of integration. These two arbitrary constants are frequently determined by the **initial position** $x_0 = x(0)$ and the **initial velocity** $v_0 = v(0)$ of the particle.

Constant acceleration. For instance, suppose that the force F, and therefore the acceleration $a = F/m$, are *constant*. Then we begin with the equation

$$\frac{dv}{dt} = a \quad (a \text{ is a constant}) \tag{9}$$

and integrate both sides to obtain

$$v(t) = \int a\,dt = at + C_1.$$

We know that $v = v_0$ when $t = 0$, and substitution of this information into the preceding equation yields the fact that $C_1 = v_0$. So

$$v(t) = \frac{dx}{dt} = at + v_0. \tag{10}$$

A second integration gives

$$x(t) = \int v(t)\,dt = \int (at + v_0)\,dt = \tfrac{1}{2}at^2 + v_0 t + C_2,$$

and the substitution $t = 0$, $x = x_0$ gives $C_2 = x_0$. Therefore,

$$x(t) = \tfrac{1}{2}at^2 + v_0 t + x_0. \tag{11}$$

Thus, with Eq. (10) we can find the velocity, and with Eq. (11) the position, of the particle at any time t in terms of its *constant* acceleration a, its initial velocity v_0, and its initial position x_0.

Example 2 A lunar lander is falling freely toward the surface of the moon at a speed of 450 meters per second (m/s). Its retrorockets, when fired, provide a constant deceleration of 2.5 meters per second per second (m/s^2) (the gravitational acceleration produced by the moon is assumed to be included in the given deceleration). At what height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown" ($v = 0$ at impact)?

Solution We denote by $x(t)$ the height of the lunar lander above the surface, as indicated in Fig. 1.2.4. We let $t = 0$ denote the time at which the retrorockets should be fired. Then $v_0 = -450$ (m/s, negative because the height $x(t)$ is decreasing), and $a = +2.5$, because an upward thrust increases the velocity v (although it decreases the *speed* $|v|$). Then Eqs. (10) and (11) become

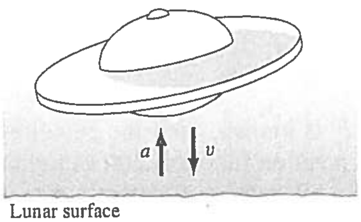

FIGURE 1.2.4. The lunar lander of Example 2.

$$v(t) = 2.5t - 450 \tag{12}$$

and

$$x(t) = 1.25t^2 - 450t + x_0, \tag{13}$$

where x_0 is the height of the lander above the lunar surface at the time $t = 0$ when the retrorockets should be activated.

From Eq. (12) we see that $v = 0$ (soft touchdown) occurs when $t = 450/2.5 = 180$ s (that is, 3 minutes); then substitution of $t = 180$, $x = 0$ into Eq. (13) yields

$$x_0 = 0 - (1.25)(180)^2 + 450(180) = 40{,}500$$

meters—that is, $x_0 = 40.5$ km $\approx 25\frac{1}{6}$ miles. Thus the retrorockets should be activated when the lunar lander is 40.5 kilometers above the surface of the moon, and it will touch down softly on the lunar surface after 3 minutes of decelerating descent. ■

Physical Units

Numerical work requires units for the measurement of physical quantities such as distance and time. We sometimes use ad hoc units—such as distance in miles or kilometers and time in hours—in special situations (such as in a problem involving an auto trip). However, the foot-pound-second (fps) and meter-kilogram-second (mks) unit systems are used more generally in scientific and engineering problems. In fact, fps units are commonly used only in the United States (and a few other countries), while mks units constitute the standard international system of scientific units.

	fps units	**mks units**
Force	pound (lb)	newton (N)
Mass	slug	kilogram (kg)
Distance	foot (ft)	meter (m)
Time	second (s)	second (s)
g	32 ft/s^2	9.8 m/s^2

The last line of this table gives values for the gravitational acceleration g at the surface of the earth. Although these approximate values will suffice for most examples and problems, more precise values are 9.7805 m/s^2 and 32.088 ft/s^2 (at sea level at the equator).

Both systems are compatible with Newton's second law $F = ma$. Thus 1 N is (by definition) the force required to impart an acceleration of 1 m/s^2 to a mass of 1 kg. Similarly, 1 slug is (by definition) the mass that experiences an acceleration of 1 ft/s^2 under a force of 1 lb. (We will use mks units in all problems requiring mass units and thus will rarely need slugs to measure mass.)

Inches and centimeters (as well as miles and kilometers) also are commonly used in describing distances. For conversions between fps and mks units it helps to remember that

$$1 \text{ in.} = 2.54 \text{ cm (exactly)} \quad \text{and} \quad 1 \text{ lb} \approx 4.448 \text{ N}.$$

For instance,

$$1 \text{ ft} = 12 \text{ in.} \times 2.54 \frac{\text{cm}}{\text{in.}} = 30.48 \text{ cm},$$

and it follows that

$$1 \text{ mi} = 5280 \text{ ft} \times 30.48 \frac{\text{cm}}{\text{ft}} = 160934.4 \text{ cm} \approx 1.609 \text{ km}.$$

Thus a posted U.S. speed limit of 50 mi/h means that—in international terms—the legal speed limit is about $50 \times 1.609 \approx 80.45$ km/h.

Vertical Motion with Gravitational Acceleration

The **weight** W of a body is the force exerted on the body by gravity. Substitution of $a = g$ and $F = W$ in Newton's second law $F = ma$ gives

$$W = mg \tag{14}$$

for the weight W of the mass m at the surface of the earth (where $g \approx 32$ ft/s$^2 \approx 9.8$ m/s^2). For instance, a mass of $m = 20$ kg has a weight of $W = (20 \text{ kg})(9.8 \text{ m/s}^2) = 196$ N. Similarly, a mass m weighing 100 pounds has mks weight

$$W = (100 \text{ lb})(4.448 \text{ N/lb}) = 444.8 \text{ N},$$

so its mass is

$$m = \frac{W}{g} = \frac{444.8 \text{ N}}{9.8 \text{ m/s}^2} \approx 45.4 \text{ kg}.$$

To discuss vertical motion it is natural to choose the y-axis as the coordinate system for position, frequently with $y = 0$ corresponding to "ground level." If we choose the *upward* direction as the positive direction, then the effect of gravity on a vertically moving body is to decrease its height and also to decrease its velocity $v = dy/dt$. Consequently, if we ignore air resistance, then the acceleration $a = dv/dt$ of the body is given by

$$\frac{dv}{dt} = -g. \tag{15}$$

This acceleration equation provides a starting point in many problems involving vertical motion. Successive integrations (as in Eqs. (10) and (11)) yield the velocity and height formulas

$$v(t) = -gt + v_0 \tag{16}$$

and

$$y(t) = -\tfrac{1}{2}gt^2 + v_0 t + y_0. \tag{17}$$

Here, y_0 denotes the initial ($t = 0$) height of the body and v_0 its initial velocity.

Example 3 **(a)** Suppose that a ball is thrown straight upward from the ground ($y_0 = 0$) with initial velocity $v_0 = 96$ (ft/s, so we use $g = 32$ ft/s^2 in fps units). Then it reaches its maximum height when its velocity (Eq. (16)) is zero,

$$v(t) = -32t + 96 = 0,$$

and thus when $t = 3$ s. Hence the maximum height that the ball attains is

$$y(3) = -\tfrac{1}{2} \cdot 32 \cdot 3^2 + 96 \cdot 3 + 0 = 144 \text{ (ft)}$$

(with the aid of Eq. (17)).
(b) If an arrow is shot straight upward from the ground with initial velocity $v_0 = 49$ (m/s, so we use $g = 9.8$ m/s^2 in mks units), then it returns to the ground when

$$y(t) = -\tfrac{1}{2} \cdot (9.8)t^2 + 49t = (4.9)t(-t + 10) = 0,$$

and thus after 10 s in the air. ■

A Swimmer's Problem

Figure 1.2.5 shows a northward-flowing river of width $w = 2a$. The lines $x = \pm a$ represent the banks of the river and the y-axis its center. Suppose that the velocity v_R at which the water flows increases as one approaches the center of the river, and indeed is given in terms of distance x from the center by

$$v_R = v_0\left(1 - \frac{x^2}{a^2}\right). \tag{18}$$

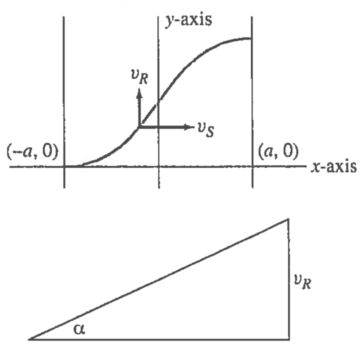

FIGURE 1.2.5. A swimmer's problem (Example 4).

You can use Eq. (18) to verify that the water does flow the fastest at the center, where $v_R = v_0$, and that $v_R = 0$ at each riverbank.

Suppose that a swimmer starts at the point $(-a, 0)$ on the west bank and swims due east (relative to the water) with constant speed v_S. As indicated in Fig. 1.2.5, his velocity vector (relative to the riverbed) has horizontal component v_S and vertical component v_R. Hence the swimmer's direction angle α is given by

$$\tan\alpha = \frac{v_R}{v_S}.$$

Because $\tan\alpha = dy/dx$, substitution using (18) gives the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right) \tag{19}$$

for the swimmer's trajectory $y = y(x)$ as he crosses the river.

Example 4 Suppose that the river is 1 mile wide and that its midstream velocity is $v_0 = 9$ mi/h. If the swimmer's velocity is $v_S = 3$ mi/h, then Eq. (19) takes the form

$$\frac{dy}{dx} = 3(1 - 4x^2).$$

Integration yields

$$y(x) = \int (3 - 12x^2)\,dx = 3x - 4x^3 + C$$

for the swimmer's trajectory. The initial condition $y\left(-\frac{1}{2}\right) = 0$ yields $C = 1$, so

$$y(x) = 3x - 4x^3 + 1.$$

Then

$$y\left(\tfrac{1}{2}\right) = 3\left(\tfrac{1}{2}\right) - 4\left(\tfrac{1}{2}\right)^3 + 1 = 2,$$

so the swimmer drifts 2 miles downstream while he swims 1 mile across the river. ■

1.2 Problems

In Problems 1 through 10, find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

1. $\dfrac{dy}{dx} = 2x + 1;\ y(0) = 3$

2. $\dfrac{dy}{dx} = (x-2)^2;\ y(2) = 1$

3. $\dfrac{dy}{dx} = \sqrt{x};\ y(4) = 0$

4. $\dfrac{dy}{dx} = \dfrac{1}{x^2};\ y(1) = 5$

5. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x+2}};\ y(2) = -1$

6. $\dfrac{dy}{dx} = x\sqrt{x^2+9};\ y(-4) = 0$

7. $\dfrac{dy}{dx} = \dfrac{10}{x^2+1};\ y(0) = 0$

8. $\dfrac{dy}{dx} = \cos 2x;\ y(0) = 1$

9. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}};\ y(0) = 0$

10. $\dfrac{dy}{dx} = xe^{-x};\ y(0) = 1$

In Problems 11 through 18, find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

11. $a(t) = 50,\ v_0 = 10,\ x_0 = 20$

12. $a(t) = -20,\ v_0 = -15,\ x_0 = 5$

13. $a(t) = 3t,\ v_0 = 5,\ x_0 = 0$

14. $a(t) = 2t + 1,\ v_0 = -7,\ x_0 = 4$

15. $a(t) = 4(t+3)^2,\ v_0 = -1,\ x_0 = 1$

16. $a(t) = \dfrac{1}{\sqrt{t+4}},\ v_0 = -1,\ x_0 = 1$

17. $a(t) = \dfrac{1}{(t+1)^3},\ v_0 = 0,\ x_0 = 0$

18. $a(t) = 50 \sin 5t,\ v_0 = -10,\ x_0 = 8$

In Problems 19 through 22, a particle starts at the origin and travels along the x-axis with the velocity function $v(t)$ whose graph is shown in Figs. 1.2.6 through 1.2.9. Sketch the graph of the resulting position function $x(t)$ for $0 \leqq t \leqq 10$.

19.

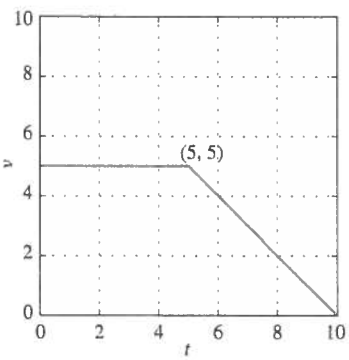

FIGURE 1.2.6. Graph of the velocity function $v(t)$ of Problem 19.

20.

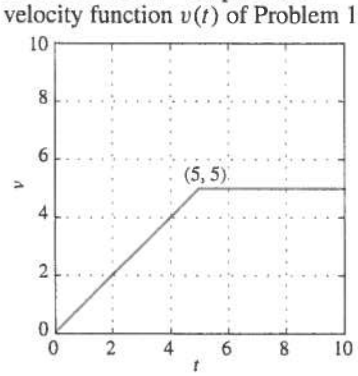

FIGURE 1.2.7. Graph of the velocity function $v(t)$ of Problem 20.

21.

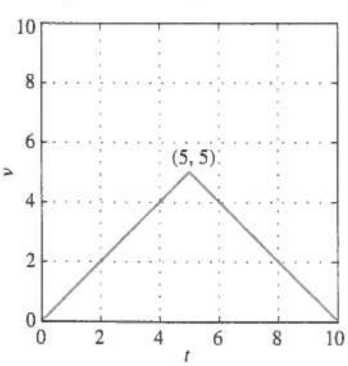

FIGURE 1.2.8. Graph of the velocity function $v(t)$ of Problem 21.

22.

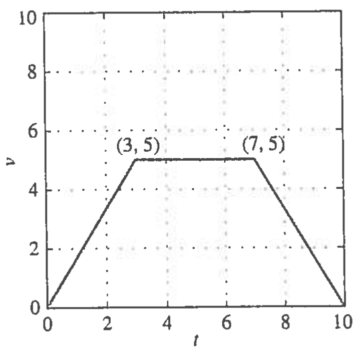

FIGURE 1.2.9. Graph of the velocity function $v(t)$ of Problem 22.

23. What is the maximum height attained by the arrow of part (b) of Example 3?

24. A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

25. The brakes of a car are applied when it is moving at 100 km/h and provide a constant deceleration of 10 meters per second per second (m/s^2). How far does the car travel before coming to a stop?

26. A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground; (b) when it passes the top of the building; (c) its total time in the air.

27. A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is 10 m/s. It strikes the ground with a speed of 60 m/s. How tall is the building?

28. A baseball is thrown straight downward with an initial speed of 40 ft/s from the top of the Washington Monument (555 ft high). How long does it take to reach the ground, and with what speed does the baseball strike the ground?

29. A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dv}{dt} = (0.12)t^2 + (0.6)t \quad (\text{ft/s}^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 s and its velocity at that time.

30. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

31. The skid marks made by an automobile indicated that its brakes were fully applied for a distance of 75 m before it came to a stop. The car in question is known to have a constant deceleration of 20 m/s^2 under these conditions. How fast—in km/h—was the car traveling when the brakes were first applied?

32. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

33. On the planet Gzyx, a ball dropped from a height of 20 ft hits the ground in 2 s. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit?

34. A person can throw a ball straight upward from the surface of the earth to a maximum height of 144 ft. How high could this person throw the ball on the planet Gzyx of Problem 33?

35. A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

36. Suppose a woman has enough "spring" in her legs to jump (on earth) from the ground to a height of 2.25 feet. If she jumps straight upward with the same initial velocity on the moon—where the surface gravitational acceleration is (approximately) 5.3 ft/s^2—how high above the surface will she rise?

37. At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from A to B?

38. At noon a car starts from rest at point A and proceeds with constant acceleration along a straight road toward point C, 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C?

39. If $a = 0.5$ mi and $v_0 = 9$ mi/h as in Example 4, what must the swimmer's speed v_S be in order that he drifts only 1 mile downstream as he crosses the river?

40. Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_S = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.

41. A bomb is dropped from a helicopter hovering at an altitude of 800 feet above the ground. From the ground directly beneath the helicopter, a projectile is fired straight upward toward the bomb, exactly 2 seconds after the bomb is released. With what initial velocity should the projectile be fired, in order to hit the bomb at an altitude of exactly 400 feet?

42. A spacecraft is in free fall toward the surface of the moon at a speed of 1000 mph (mi/h). Its retrorockets, when fired, provide a constant deceleration of 20,000 mi/h^2. At what height above the lunar surface should the astronauts fire the retrorockets to insure a soft touchdown? (As in Example 2, ignore the moon's gravitational field.)

43. Arthur Clarke's *The Wind from the Sun* (1963) describes Diana, a spacecraft propelled by the solar wind. Its aluminized sail provides it with a constant acceleration of $0.001g = 0.0098 \text{ m/s}^2$. Suppose this spacecraft starts from rest at time $t = 0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth of the speed $c = 3 \times 10^8$ m/s of light. How long will it take the spacecraft to catch up with the projectile, and how far will it have traveled by then?

44. A driver involved in an accident claims he was going only 25 mph. When police tested his car, they found that when its brakes were applied at 25 mph, the car skidded only 45 feet before coming to a stop. But the driver's skid marks at the accident scene measured 210 feet. Assuming the same (constant) deceleration, determine the speed he was actually traveling just prior to the accident.

1.3 Slope Fields and Solution Curves

Consider a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

where the right-hand function $f(x, y)$ involves both the independent variable x and the dependent variable y. We might think of integrating both sides in (1) with respect to x, and hence write $y(x) = \int f(x, y(x))\,dx + C$. However, this approach does not lead to a solution of the differential equation, because the indicated integral involves the *unknown* function $y(x)$ itself, and therefore cannot be evaluated explicitly. Actually, there exists *no* straightforward procedure by which a general differential equation can be solved explicitly. Indeed, the solutions of such a simple-looking differential equation as $y' = x^2 + y^2$ cannot be expressed in terms of the ordinary elementary functions studied in calculus textbooks. Nevertheless, the graphical and numerical methods of this and later sections can be used to construct *approximate* solutions of differential equations that suffice for many practical purposes.

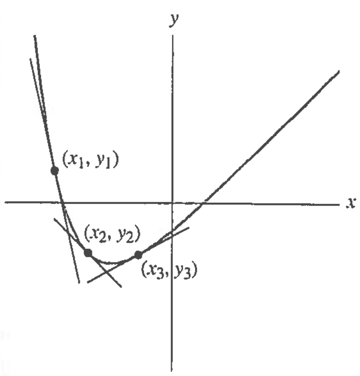

FIGURE 1.3.1. A solution curve for the differential equation $y' = x - y$ together with tangent lines having

- slope $m_1 = x_1 - y_1$ at the point (x_1, y_1);
- slope $m_2 = x_2 - y_2$ at the point (x_2, y_2); and
- slope $m_3 = x_3 - y_3$ at the point (x_3, y_3).

Slope Fields and Graphical Solutions

There is a simple geometric way to think about solutions of a given differential equation $y' = f(x, y)$. At each point (x, y) of the xy-plane, the value of $f(x, y)$ determines a slope $m = f(x, y)$. A solution of the differential equation is simply a differentiable function whose graph $y = y(x)$ has this "correct slope" at each point $(x, y(x))$ through which it passes—that is, $y'(x) = f(x, y(x))$. Thus a **solution curve** of the differential equation $y' = f(x, y)$—the graph of a solution of the equation—is simply a curve in the xy-plane whose tangent line at each point (x, y) has slope $m = f(x, y)$. For instance, Fig. 1.3.1 shows a solution curve of the differential equation $y' = x - y$ together with its tangent lines at three typical points.

This geometric viewpoint suggests a *graphical method* for constructing *approximate* solutions of the differential equation $y' = f(x, y)$. Through each of a representative collection of points (x, y) in the plane we draw a short line segment having the proper slope $m = f(x, y)$. All these line segments constitute a **slope field** (or a **direction field**) for the equation $y' = f(x, y)$.

Example 1 Figures 1.3.2 (a)–(d) show slope fields and solution curves for the differential equation

$$\frac{dy}{dx} = ky \tag{2}$$

with the values $k = 2, 0.5, -1$, and -3 of the parameter k in Eq. (2). Note that each slope field yields important qualitative information about the set of all solutions

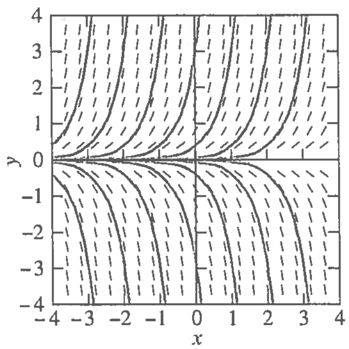

FIGURE 1.3.2(a) Slope field and solution curves for $y' = 2y$.

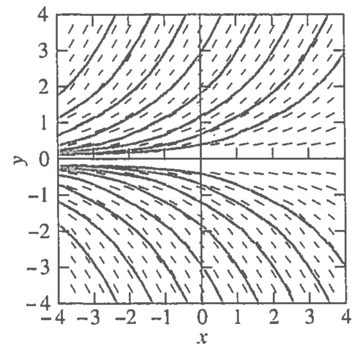

FIGURE 1.3.2(b) Slope field and solution curves for $y' = (0.5)y$.

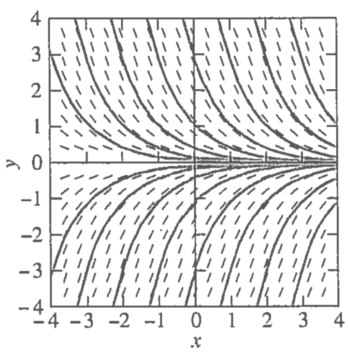

FIGURE 1.3.2(c) Slope field and solution curves for $y' = -y$.

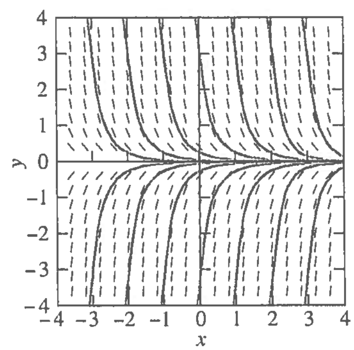

FIGURE 1.3.2(d) Slope field and solution curves for $y' = -3y$.

of the differential equation. For instance, Figs. 1.3.2(a) and (b) suggest that each solution $y(x)$ approaches $\pm\infty$ as $x \to +\infty$ if $k > 0$, whereas Figs. 1.3.2(c) and (d) suggest that $y(x) \to 0$ as $x \to +\infty$ if $k < 0$. Moreover, although the sign of k determines the *direction* of increase or decrease of $y(x)$, its absolute value $|k|$ appears to determine the *rate of change* of $y(x)$. All this is apparent from slope fields like those in Fig. 1.3.2, even without knowing that the general solution of Eq. (2) is given explicitly by $y(x) = Ce^{kx}$. ■

A slope field suggests visually the general shapes of solution curves of the differential equation. Through each point a solution curve should proceed in such a direction that its tangent line is nearly parallel to the nearby line segments of the slope field. Starting at any initial point (a, b), we can attempt to sketch freehand an approximate solution curve that threads its way through the slope field, following the visible line segments as closely as possible.

Example 2 Construct a slope field for the differential equation $y' = x - y$ and use it to sketch an approximate solution curve that passes through the point $(-4, 4)$.

Solution Figure 1.3.3 shows a table of slopes for the given equation. The numerical slope $m = x - y$ appears at the intersection of the horizontal x-row and the vertical y-column of the table. If you inspect the pattern of upper-left to lower-right diagonals in this table, you can see that it was easily and quickly constructed. (Of

$x \backslash y$	−4	−3	−2	−1	0	1	2	3	4
−4	0	−1	−2	−3	−4	−5	−6	−7	−8
−3	1	0	−1	−2	−3	−4	−5	−6	−7
−2	2	1	0	−1	−2	−3	−4	−5	−6
−1	3	2	1	0	−1	−2	−3	−4	−5
0	4	3	2	1	0	−1	−2	−3	−4
1	5	4	3	2	1	0	−1	−2	−3
2	6	5	4	3	2	1	0	−1	−2
3	7	6	5	4	3	2	1	0	−1
4	8	7	6	5	4	3	2	1	0

FIGURE 1.3.3. Values of the slope $y' = x - y$ for $-4 \leqq x,\ y \leqq 4$.

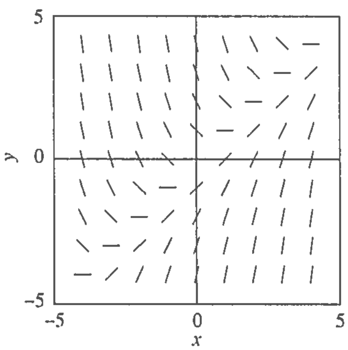

FIGURE 1.3.4. Slope field for $y' = x - y$ corresponding to the table of slopes in Fig. 1.3.3.

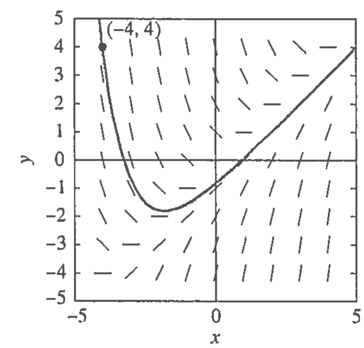

FIGURE 1.3.5. The solution curve through $(-4, 4)$.

course, a more complicated function $f(x, y)$ on the right-hand side of the differential equation would necessitate more complicated calculations.) Figure 1.3.4 shows the corresponding slope field, and Fig. 1.3.5 shows an approximate solution curve sketched through the point $(-4, 4)$ so as to follow this slope field as closely as possible. At each point it appears to proceed in the direction indicated by the nearby line segments of the slope field. ■

Although a spreadsheet program (for instance) readily constructs a table of slopes as in Fig. 1.3.3, it can be quite tedious to plot by hand a sufficient number of slope segments as in Fig. 1.3.4. However, most computer algebra systems include commands for quick and ready construction of slope fields with as many line segments as desired; such commands are illustrated in the application material for this section. The more line segments are constructed, the more accurately solution curves can be visualized and sketched. Figure 1.3.6 shows a "finer" slope field for the differential equation $y' = x - y$ of Example 2, together with typical solution curves treading through this slope field.

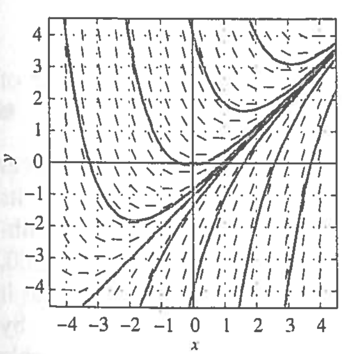

FIGURE 1.3.6. Slope field and typical solution curves for $y' = x - y$.

If you look closely at Fig. 1.3.6, you may spot a solution curve that appears to be a straight line! Indeed, you can verify that the linear function $y = x - 1$ is a solution of the equation $y' = x - y$, and it appears likely that the other solution curves approach this straight line as an asymptote as $x \to +\infty$. This inference illustrates the fact that a slope field can suggest tangible information about solutions that is not at all evident from the differential equation itself. Can you, by tracing the appropriate solution curve in this figure, infer that $y(3) \approx 2$ for the solution $y(x)$ of the initial value problem $y' = x - y$, $y(-4) = 4$?

Applications of Slope Fields

The next two examples illustrate the use of slope fields to glean useful information in physical situations that are modeled by differential equations. Example 3 is based on the fact that a baseball moving through the air at a moderate speed v (less than about 300 ft/s) encounters air resistance that is approximately proportional to v. If the baseball is thrown straight downward from the top of a tall building or from a hovering helicopter, then it experiences both the downward acceleration of gravity and an upward acceleration of air resistance. If the y-axis is directed *downward*, then the ball's velocity $v = dy/dt$ and its gravitational acceleration $g = 32$ ft/s^2 are both positive, while its acceleration due to air resistance is negative. Hence its total acceleration is of the form

$$\frac{dv}{dt} = g - kv. \tag{3}$$

A typical value of the air resistance proportionality constant might be $k = 0.16$.

Example 3 Suppose you throw a baseball straight downward from a helicopter hovering at an altitude of 3000 feet. You wonder whether someone standing on the ground below could conceivably catch it. In order to estimate the speed with which the ball will land, you can use your laptop's computer algebra system to construct a slope field for the differential equation

$$\frac{dv}{dt} = 32 - 0.16v. \tag{4}$$

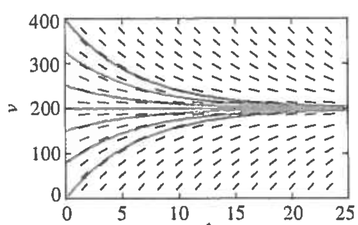

FIGURE 1.3.7. Slope field and typical solution curves for $v' = 32 - 0.16v$.

The result is shown in Fig. 1.3.7, together with a number of solution curves corresponding to different values of the initial velocity $v(0)$ with which you might throw the baseball downward. Note that all these solution curves appear to approach the horizontal line $v = 200$ as an asymptote. This implies that—however you throw it—the baseball should approach the *limiting velocity* $v = 200$ ft/s instead of accelerating indefinitely (as it would in the absence of any air resistance). The handy fact that 60 mi/h = 88 ft/s yields

$$v = 200\frac{\text{ft}}{\text{s}} \times \frac{60 \text{ mi/h}}{88 \text{ ft/s}} \approx 136.36\,\frac{\text{mi}}{\text{h}}.$$

Perhaps a catcher accustomed to 100 mi/h fastballs would have some chance of fielding this speeding ball. ■

Comment: If the ball's initial velocity is $v(0) = 200$, then Eq. (4) gives $v'(0) = 32 - (0.16)(200) = 0$, so the ball experiences *no* initial acceleration. Its velocity therefore remains unchanged, and hence $v(t) \equiv 200$ is a constant "equilibrium solution" of the differential equation. If the initial velocity is greater than 200, then the initial acceleration given by Eq. (4) is negative, so the ball slows down as it falls. But if the initial velocity is less than 200, then the initial acceleration given by (4) is positive, so the ball speeds up as it falls. It therefore seems quite reasonable that, because of air resistance, the baseball will approach a limiting velocity of 200 ft/s—whatever initial velocity it starts with. You might like to verify that—in the absence of air resistance—this ball would hit the ground at over 300 mi/h. ■

In Section 2.1 we will discuss in detail the logistic differential equation

$$\frac{dP}{dt} = kP(M - P) \tag{5}$$

that often is used to model a population $P(t)$ that inhabits an environment with *carrying capacity* M. This means that M is the maximum population that this environment can sustain on a long-term basis (in terms of the maximum available food, for instance).

Example 4 If we take $k = 0.0004$ and $M = 150$, then the logistic equation in (5) takes the form

$$\frac{dP}{dt} = 0.0004P(150 - P) = 0.06P - 0.0004P^2. \tag{6}$$

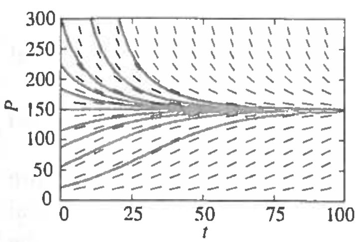

FIGURE 1.3.8. Slope field and typical solution curves for $P' = 0.06P - 0.0004P^2$.

The positive term $0.06P$ on the right in (6) corresponds to natural growth at a 6% annual rate (with time t measured in years). The negative term $-0.0004P^2$ represents the inhibition of growth due to limited resources in the environment.

Figure 1.3.8 shows a slope field for Eq. (6), together with a number of solution curves corresponding to possible different values of the initial population $P(0)$. Note that all these solution curves appear to approach the horizontal line $P = 150$ as an asymptote. This implies that—whatever the initial population—the population $P(t)$ approaches the *limiting population* $P = 150$ as $t \to \infty$. ■

Comment: If the initial population is $P(0) = 150$, then Eq. (6) gives

$$P'(0) = 0.0004(150)(150 - 150) = 0,$$

so the population experiences *no* initial (instantaneous) change. It therefore remains unchanged, and hence $P(t) \equiv 150$ is a constant "equilibrium solution" of the differential equation. If the initial population is greater than 150, then the initial rate of change given by (6) is negative, so the population immediately begins to decrease. But if the initial population is less than 150, then the initial rate of change given by (6) is positive, so the population immediately begins to increase. It therefore seems quite reasonable to conclude that the population will approach a limiting value of 150—whatever the (positive) initial population. ■

Existence and Uniqueness of Solutions

Before one spends much time attempting to solve a given differential equation, it is wise to know that solutions actually *exist.* We may also want to know whether there is only one solution of the equation satisfying a given initial condition—that is, whether its solutions are *unique.*

Example 5 **(a)** [Failure of existence] The initial value problem

$$y' = \frac{1}{x}, \quad y(0) = 0 \tag{7}$$

has *no* solution, because no solution $y(x) = \int (1/x)\,dx = \ln|x| + C$ of the differential equation is defined at $x = 0$. We see this graphically in Fig. 1.3.9, which shows a direction field and some typical solution curves for the equation $y' = 1/x$. It is apparent that the indicated direction field "forces" all solution curves near the y-axis to plunge downward so that none can pass through the point $(0, 0)$.

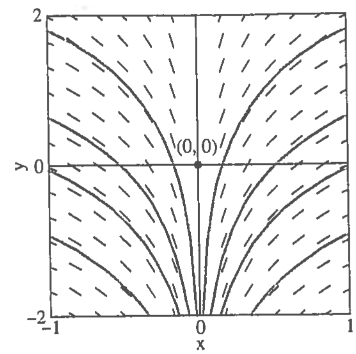

FIGURE 1.3.9. Direction field and typical solution curves for the equation $y' = 1/x$.

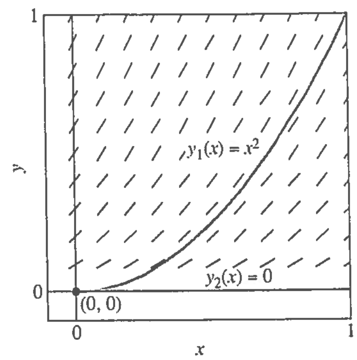

FIGURE 1.3.10. Direction field and two different solution curves for the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$.

(b) [Failure of uniqueness] On the other hand, you can readily verify that the initial value problem

$$y' = 2\sqrt{y}, \quad y(0) = 0 \tag{8}$$

has the *two* different solutions $y_1(x) = x^2$ and $y_2(x) \equiv 0$ (see Problem 27). Figure 1.3.10 shows a direction field and these two different solution curves for the initial value problem in (8). We see that the curve $y_1(x) = x^2$ threads its way through the indicated direction field, whereas the differential equation $y' = 2\sqrt{y}$ specifies slope $y' = 0$ along the x-axis $y_2(x) = 0$. ■

Example 5 illustrates the fact that, before we can speak of "the" solution of an initial value problem, we need to know that it has *one and only one* solution. Questions of existence and uniqueness of solutions also bear on the process of mathematical modeling. Suppose that we are studying a physical system whose behavior is completely determined by certain initial conditions, but that our proposed mathematical model involves a differential equation *not* having a unique solution satisfying those conditions. This raises an immediate question as to whether the mathematical model adequately represents the physical system.

The theorem stated below implies that the initial value problem $y' = f(x, y)$, $y(a) = b$ has one and only one solution defined near the point $x = a$ on the x-axis, provided that both the function f and its partial derivative $\partial f/\partial y$ are continuous near the point (a, b) in the xy-plane. Methods of proving existence and uniqueness theorems are discussed in Appendix A.

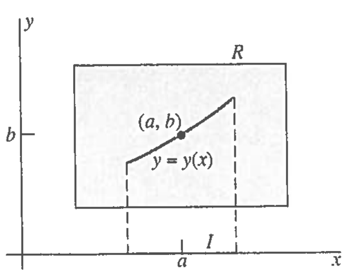

FIGURE 1.3.11. The rectangle R and x-interval I of Theorem 1, and the solution curve $y = y(x)$ through the point (a, b).

THEOREM 1 Existence and Uniqueness of Solutions

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b \tag{9}$$

has one and only one solution that is defined on the interval I. (As illustrated in Fig. 1.3.11, the solution interval I may not be as "wide" as the original rectangle R of continuity; see Remark 3 on the next page.)

Remark 1: In the case of the differential equation $dy/dx = -y$ of Example 1 and Fig. 1.3.2(c), both the function $f(x, y) = -y$ and the partial derivative $\partial f/\partial y = -1$ are continuous everywhere, so Theorem 1 implies the existence of a unique solution for any initial data (a, b). Although the theorem ensures existence only on some open interval containing $x = a$, each solution $y(x) = Ce^{-x}$ actually is defined for all x.

Remark 2: In the case of the differential equation $dy/dx = 2\sqrt{y}$ of Example 5(b) and Eq. (8), the function $f(x, y) = 2\sqrt{y}$ is continuous wherever $y > 0$, but the partial derivative $\partial f/\partial y = 1/\sqrt{y}$ is discontinuous when $y = 0$, and hence at the point (0, 0). This is why it is possible for there to exist two different solutions $y_1(x) = x^2$ and $y_2(x) \equiv 0$, each of which satisfies the initial condition $y(0) = 0$.

Remark 3: In Example 7 of Section 1.1 we examined the especially simple differential equation $dy/dx = y^2$. Here we have $f(x, y) = y^2$ and $\partial f/\partial y = 2y$. Both of these functions are continuous everywhere in the xy-plane, and in particular on the rectangle $-2 < x < 2$, $0 < y < 2$. Because the point (0, 1) lies in the interior of this rectangle, Theorem 1 guarantees a unique solution—necessarily a continuous function—of the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(0) = 1 \tag{10}$$

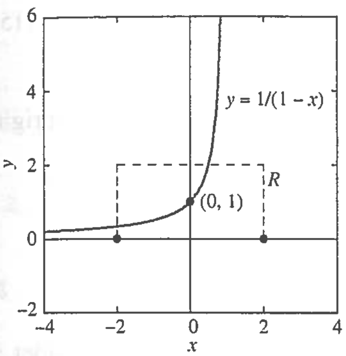

FIGURE 1.3.12. The solution curve through the initial point (0, 1) leaves the rectangle R before it reaches the right side of R.

on some open x-interval containing $a = 0$. Indeed this is the solution

$$y(x) = \frac{1}{1-x}$$

that we discussed in Example 7. But $y(x) = 1/(1-x)$ is discontinuous at $x = 1$, so our unique continuous solution does not exist on the entire interval $-2 < x < 2$. Thus the solution interval I of Theorem 1 may not be as wide as the rectangle R where f and $\partial f/\partial y$ are continuous. Geometrically, the reason is that the solution curve provided by the theorem may leave the rectangle—wherein solutions of the differential equation are guaranteed to exist—before it reaches the one or both ends of the interval (see Fig. 1.3.12). ■

The following example shows that, if the function $f(x, y)$ and/or its partial derivative $\partial f/\partial y$ fail to satisfy the continuity hypothesis of Theorem 1, then the initial value problem in (9) may have *either* no solution *or* many—even infinitely many—solutions.

Example 6 Consider the first-order differential equation

$$x\frac{dy}{dx} = 2y. \tag{11}$$

Applying Theorem 1 with $f(x, y) = 2y/x$ and $\partial f/\partial y = 2/x$, we conclude that Eq. (11) must have a unique solution near any point in the xy-plane where $x \neq 0$. Indeed, we see immediately by substitution in (11) that

$$y(x) = Cx^2 \tag{12}$$

satisfies Eq. (11) for any value of the constant C and for all values of the variable x. In particular, the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(0) = 0 \tag{13}$$

has infinitely many different solutions, whose solution curves are the parabolas $y = Cx^2$ illustrated in Fig. 1.3.13. (In case $C = 0$, the "parabola" is actually the x-axis $y = 0$.)

Observe that all these parabolas pass through the origin $(0, 0)$, but none of them passes through any other point on the y-axis. It follows that the initial value problem in (13) has infinitely many solutions, but the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(0) = b \tag{14}$$

has no solution if $b \neq 0$.

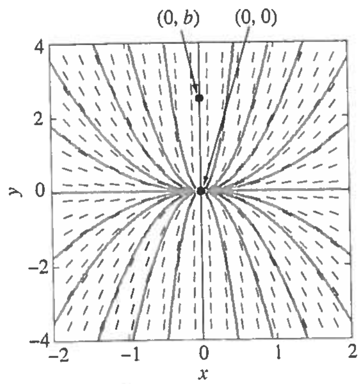

FIGURE 1.3.13. There are infinitely many solution curves through the point $(0, 0)$, but no solution curves through the point $(0, b)$ if $b \neq 0$.

Finally, note that through any point off the y-axis there passes only one of the parabolas $y = Cx^2$. Hence, if $a \neq 0$, then the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(a) = b \tag{15}$$

has a unique solution on any interval that contains the point $x = a$ but not the origin $x = 0$. In summary, the initial value problem in (15) has

- a unique solution near (a, b) if $a \neq 0$;
- no solution if $a = 0$ but $b \neq 0$;
- infinitely many solutions if $a = b = 0$.

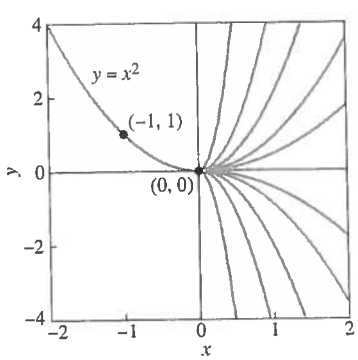

FIGURE 1.3.14. There are infinitely many solution curves through the point $(1, -1)$.

Still more can be said about the initial value problem in (15). Consider a typical initial point off the y-axis—for instance, the point $(-1, 1)$ indicated in Fig. 1.3.14. Then for any value of the constant C the function defined by

$$y(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ Cx^2 & \text{if } x > 0 \end{cases} \tag{16}$$

is continuous and satisfies the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(-1) = 1. \tag{17}$$

For a particular value of C, the solution curve defined by (16) consists of the left half of the parabola $y = x^2$ and the right half of the parabola $y = Cx^2$. Thus the unique solution curve near $(-1, 1)$ branches at the origin into the infinitely many solution curves illustrated in Fig. 1.3.14.

We therefore see that Theorem 1 (if its hypotheses are satisfied) guarantees uniqueness of the solution near the initial point (a, b), but a solution curve through (a, b) may eventually branch elsewhere so that uniqueness is lost. Thus a solution may exist on a larger interval than one on which the solution is unique. For instance, the solution $y(x) = x^2$ of the initial value problem in (17) exists on the whole x-axis, but this solution is unique only on the negative x-axis $-\infty < x < 0$.

1.3 Problems

In Problems 1 through 10, we have provided the slope field of the indicated differential equation, together with one or more solution curves. Sketch likely solution curves through the additional points marked in each slope field.

1. $\dfrac{dy}{dx} = -y - \sin x$

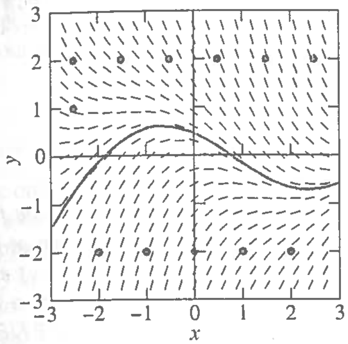

FIGURE 1.3.15

2. $\dfrac{dy}{dx} = x + y$

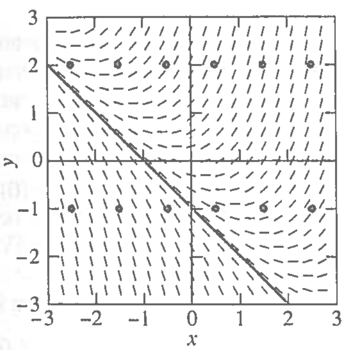

FIGURE 1.3.16

3. $\dfrac{dy}{dx} = y - \sin x$

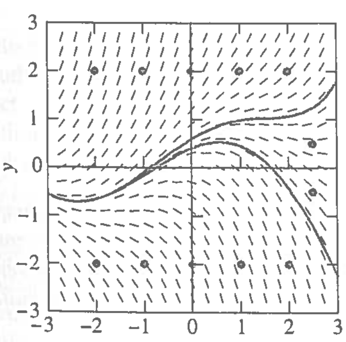

FIGURE 1.3.17

4. $\dfrac{dy}{dx} = x - y$

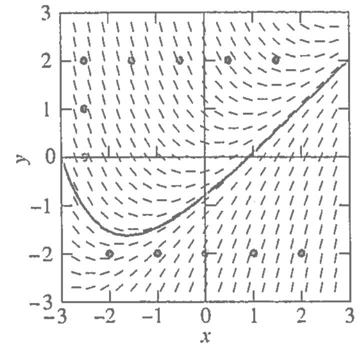

FIGURE 1.3.18

5. $\dfrac{dy}{dx} = y - x + 1$

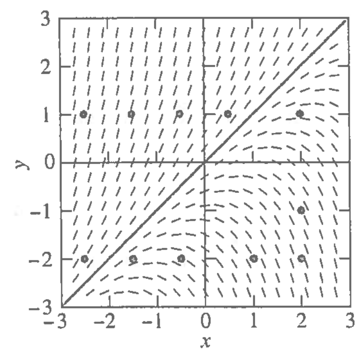

FIGURE 1.3.19

6. $\dfrac{dy}{dx} = x - y + 1$

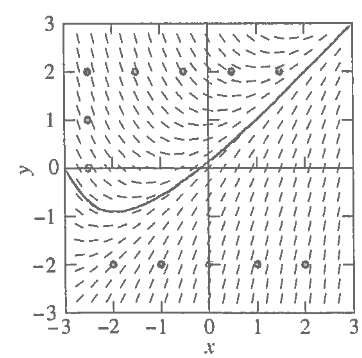

FIGURE 1.3.20

7. $\dfrac{dy}{dx} = \sin x + \sin y$

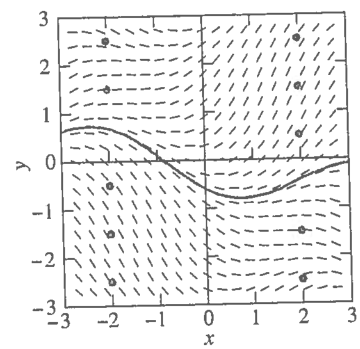

FIGURE 1.3.21

8. $\dfrac{dy}{dx} = x^2 - y$

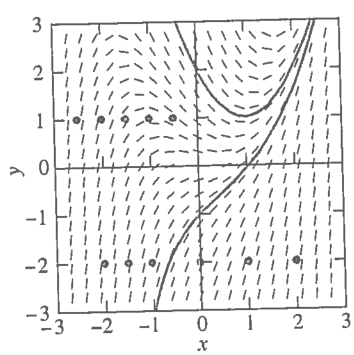

FIGURE 1.3.22

9. $\dfrac{dy}{dx} = x^2 - y - 2$

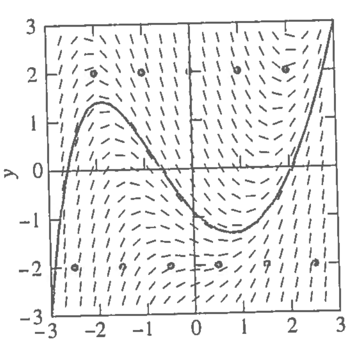

FIGURE 1.3.23

10. $\dfrac{dy}{dx} = -x^2 + \sin y$

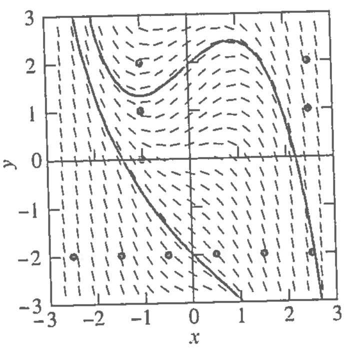

FIGURE 1.3.24

A more detailed version of Theorem 1 says that, if the function $f(x, y)$ is continuous near the point (a, b), then at least one solution of the differential equation $y' = f(x, y)$ exists on some open interval I containing the point $x = a$ and, moreover, that if in addition the partial derivative $\partial f/\partial y$ is continuous near (a, b), then this solution is unique on some (perhaps smaller) interval J. In Problems 11 through 20, determine whether existence of at least one solution of the given initial value problem is thereby guaranteed and, if so, whether uniqueness of that solution is guaranteed.

11. $\dfrac{dy}{dx} = 2x^2y^2; \quad y(1) = -1$

12. $\dfrac{dy}{dx} = x \ln y; \quad y(1) = 1$

13. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 1$

14. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 0$

15. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 2$

16. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 1$

17. $y\dfrac{dy}{dx} = x - 1; \quad y(0) = 1$

18. $y\dfrac{dy}{dx} = x - 1; \quad y(1) = 0$

19. $\dfrac{dy}{dx} = \ln(1 + y^2); \quad y(0) = 0$

20. $\dfrac{dy}{dx} = x^2 - y^2; \quad y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y, \quad y(0) = 0; \quad y(-4) = ?$
22. $y' = y - x, \quad y(4) = 0; \quad y(-4) = ?$

Problems 23 and 24 are like Problems 21 and 22, but now use a computer algebra system to plot and print out a slope field for the given differential equation. If you wish (and know how), you can check your manually sketched solution curve by plotting it with the computer.

23. $y' = x^2 + y^2 - 1, \quad y(0) = 0; \quad y(2) = ?$

24. $y' = x + \frac{1}{2}y^2, \quad y(-2) = 0; \quad y(2) = ?$

25. You bail out of the helicopter of Example 3 and pull the ripcord of your parachute. Now $k = 1.6$ in Eq. (3), so your downward velocity satisfies the initial value problem

$$\frac{dv}{dt} = 32 - 1.6v, \quad v(0) = 0.$$

In order to investigate your chances of survival, construct a slope field for this differential equation and sketch the appropriate solution curve. What will your limiting velocity be? Will a strategically located haystack do any good? How long will it take you to reach 95% of your limiting velocity?

26. Suppose the deer population $P(t)$ in a small forest satisfies the logistic equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2.$$

Construct a slope field and appropriate solution curve to answer the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take the number of deer to double? What will be the limiting deer population?

The next seven problems illustrate the fact that, if the hypotheses of Theorem 1 are not satisfied, then the initial value problem $y' = f(x, y)$, $y(a) = b$ may have either no solutions, finitely many solutions, or infinitely many solutions.

27. **(a)** Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x-c)^2 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 2\sqrt{y}$ for all x (including the point $x = c$). Construct a figure illustrating the fact that the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has infinitely many different solutions. **(b)** For what values of b does the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ have (i) no solution, (ii) a unique solution that is defined for all x?

28. Verify that if k is a constant, then the function $y(x) \equiv kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y$, $y(a) = b$ has—one, none, or infinitely many.

29. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x-c)^3 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 3y^{2/3}$ for all x. Can you also use the "left half" of the cubic $y = (x-c)^3$ in piecing together a solution curve of the differential equation? (See Fig. 1.3.25.) Sketch a variety of such solution curves. Is there a point (a, b) of the xy-plane such that the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ has either no solution or a unique solution that is defined for all x? Reconcile your answer with Theorem 1.

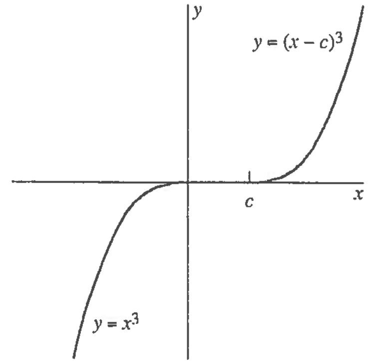

FIGURE 1.3.25. A suggestion for Problem 29.

30. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} +1 & \text{if } x \leqq c, \\ \cos(x-c) & \text{if } c < x < c + \pi, \\ -1 & \text{if } x \geqq c + \pi \end{cases}$$

satisfies the differential equation $y' = -\sqrt{1-y^2}$ for all x. (Perhaps a preliminary sketch with $c = 0$ will be helpful.) Sketch a variety of such solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has.

31. Carry out an investigation similar to that in Problem 30, except with the differential equation $y' = +\sqrt{1-y^2}$. Does it suffice simply to replace $\cos(x-c)$ with $\sin(x-c)$ in piecing together a solution that is defined for all x?

32. Verify that if $c > 0$, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{if } x^2 \leqq c, \\ (x^2-c)^2 & \text{if } x^2 > c \end{cases}$$

satisfies the differential equation $y' = 4x\sqrt{y}$ for all x. Sketch a variety of such solution curves for different values of c. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has.

33. If $c \neq 0$, verify that the function defined by $y(x) = x/(cx - 1)$ (with graph illustrated in Fig. 1.3.26) satisfies the differential equation $x^2y' + y^2 = 0$ if $x \neq 1/c$. Sketch a variety of such solution curves for different values of c. Also, note the constant-valued function $y(x) \equiv 0$ that does not result from any choice of the constant c. Finally, determine (in terms of a and b) how many different solutions the initial value problem $x^2y' + y^2 = 0$, $y(a) = b$ has.

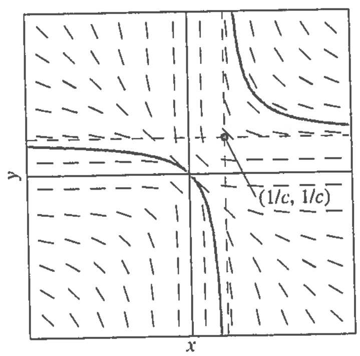

FIGURE 1.3.26. Slope field for $x^2y' + y^2 = 0$ and graph of a solution $y(x) = x/(cx - 1)$.

34. (a) Use the direction field of Problem 5 to estimate the values at $x = 1$ of the two solutions of the differential equation $y' = y - x + 1$ with initial values $y(-1) = -1.2$ and $y(-1) = -0.8$.

(b) Use a computer algebra system to estimate the values at $x = 3$ of the two solutions of this differential equation with initial values $y(-3) = -3.01$ and $y(-3) = -2.99$.

The lesson of this problem is that small changes in initial conditions can make big differences in results.

35. (a) Use the direction field of Problem 6 to estimate the values at $x = 2$ of the two solutions of the differential equation $y' = x - y + 1$ with initial values $y(-3) = -0.2$ and $y(-3) = +0.2$.

(b) Use a computer algebra system to estimate the values at $x = 2$ of the two solutions of this differential equation with initial values $y(-3) = -0.5$ and $y(-3) = +0.5$.

The lesson of this problem is that big changes in initial conditions may make only small differences in results.

1.3 Application Computer-Generated Slope Fields and Solution Curves

Widely available computer algebra systems and technical computing environments include facilities to automate the construction of slope fields and solution curves, as do some graphing calculators (see Fig. 1.3.27).

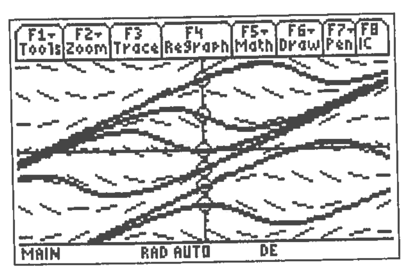

FIGURE 1.3.27. Slope field and solution curves for the differential equation

$$\frac{dy}{dx} = \sin(x - y)$$

with initial points $(0, b)$, $b = -3, -1, -2, 0, 2, 4$ and window $-5 \leq x, y \leq 5$ on a TI-89 graphing calculator.

The applications manual accompanying this textbook includes discussion of *Maple*™, *Mathematica*™, and MATLAB™ resources for the investigation of differential equations. For instance, the *Maple* command

```
with(DEtools):
DEplot(diff(y(x),x)=sin(x-y(x)), y(x), x=-5..5, y=-5..5);
```