Math 2250-1
Tues Sept 4
2.1-2.2 Autonomous differential equations

Final intro to Maple sessions in LCB 115:
Tuesday 2-2:50 PM
Wednesday 10:45-11:35 AM, 3:05-3:55 PM
Thursday 11:50 AM-12:40 PM

• Discuss the improved population models in section 2.1, in particular the logistic model. This will take a large part of today's presentation, and will use last Friday's notes.

Then begin
2.2: Autonomous Differential Equations.
Recall, that if we solve for the derivative, a general first order DE for $x = x(t)$ is written as

$$x' = f(t, x) \,,$$

which is shorthand for $x'(t) = f(t, x(t))$.
Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) \,.$$

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$, for example.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of
1a) $x'(t) = 3\,x - x^2$

1b) $x'(t) = x^3 + 2\,x^2 + x$

1c) $x'(t) = \sin(x)$.

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c. There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$

2b) $x'(t) = x^3 + 2x^2 + x$

2c) $x'(t) = \sin(x)$.

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

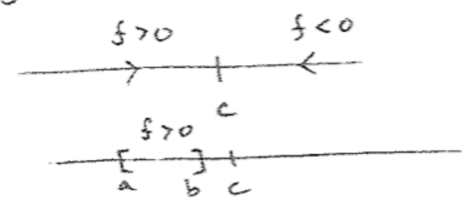
with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$sign(f)$: $- - - - 0 + + +$ $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable
$sign(f)$: $+ + + 0 - - - -$ $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is asymptotically stable
$sign(f)$: $+ + + 0 + + +$ $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable (half stable)
$sign(f)$: $- - - - 0 - - - -$ $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is unstable (half stable)

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; $f>0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

(extreme value thm from calculus, f attains its minimum).

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$ ∎