Math 2250-1
Wed Sept 19
3.4 Matrix algebra

• Our first exam is next Thursday Sept. 27. You have a homework assignment covering 3.5-3.6 which is due Wednesday Sept. 26, and which is posted on our homework page. (The 3.1-3.4 homework is due this Friday.) The exam will cover through 3.6.

• Today we first discuss the general conclusions about how the shape of the reduced row echelon form of a matrix A influences the possible solution sets to linear systems of equations for which it is the coefficient matrix. This is Exercise 6 on Tuesday's notes, which you were asked to think about before class today.

....

• Then we will discuss vector and matrix algebra, section 3.4.

Matrix vector algebra that we've already touched on, and discussed yesterday:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ : \\ x_n + y_n \end{bmatrix} ; \qquad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ : \\ c\,x_n \end{bmatrix}$$

Vector dot product , which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := x_1\,y_1 + x_2\,y_2 + ... + x_n\,y_n .$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{\boldsymbol{x}}$ is an n column vector, then

$$A\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2\,n} \\ : & & & & : \\ a_{m1} & a_{m2} & a_{m3} & ... & a_{m\,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1\,n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2\,n}x_n \\ : \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{m\,n}x_n \end{bmatrix} = \begin{bmatrix} Row_1(A) \cdot \underline{\boldsymbol{x}} \\ Row_2(A) \cdot \underline{\boldsymbol{x}} \\ : \\ Row_m(A) \cdot \underline{\boldsymbol{x}} \end{bmatrix}$$

Compact way to write our usual linear system:

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}} .$$

Exercise 1: Check that matrix multiplication distributes over vector addition and scalar multiplication, i.e.

$$A(\underline{\mathbf{x}} + \underline{\mathbf{y}}) = A\,\underline{\mathbf{x}} + A\,\underline{\mathbf{y}}$$
$$A(c\,\underline{\mathbf{x}}) = c\,A\,\underline{\mathbf{x}}$$

Exercise 2 (relates to hw and an example from yesterday): Consider the matrix equation

$$A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}\,.$$

Assume the system is consistent, so that there is at least one (particular) solution $\underline{\boldsymbol{x}}_P$, i.e. a vector $\underline{\boldsymbol{x}}_P$ so that

$$A\,\underline{\boldsymbol{x}}_P = \underline{\boldsymbol{b}}.$$

Use the algebra properties of the previous page to show that

2a) If $\underline{\boldsymbol{x}}_H$ solves the homogeneous system $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$, then $\underline{\mathbf{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ solves the original problem

$$A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}\,.$$

2b) Show that *every* solution to $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is of the form

$$\underline{\mathbf{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$$

where $\underline{\boldsymbol{x}}_P$ is the same single particular solution, and $\underline{\boldsymbol{x}}_H$ is *some* solution to the homogeneous equation.

Matrix algebra:

- addition and scalar multiplication: Let $A_{m \times n}$, $B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $entry_{ij}(A) = a_{ij}$, $entry_{ij}(B) = b_{ij}$. (In this case we write $A = [a_{ij}]$, $B = [b_{ij}]$.) Let c be a scalar. Then

$$entry_{ij}(A + B) := a_{ij} + b_{ij}.$$
$$entry_{ij}(c\,A) := c\,a_{ij}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

Exercise 3) Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4\,A - B$.

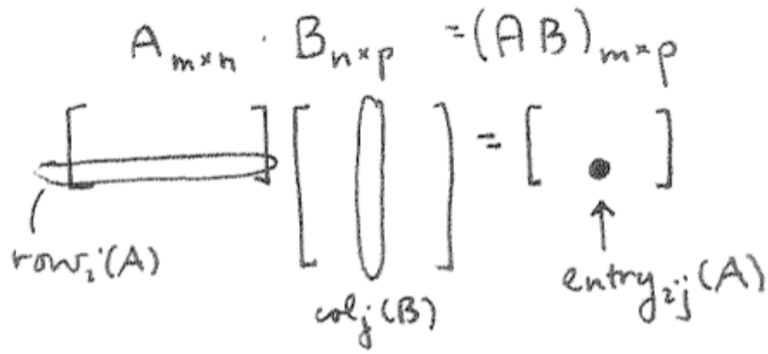
- <u>matrix multiplication:</u> Let $A_{m \times n}$, $B_{n \times p}$ be two matrices such that the number of columns of A equals the number of rows of B. Then the product AB is an $m \times p$ matrix, with

$$entry_{ij}(AB) := row_i(A) \cdot col_j(B) = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

Equivalently, the j^{th} column of AB is given by the matrix times vector product

$$col_j(AB) = A\, col_j(B) .$$

This stencil might help:



<u>Exercise 4)</u>

<u>a)</u> Can you compute AB for the matrices A, B in exercise 3?

b) Let $C := \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Using $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ compute AC and CA . Huh?

Properties for the algebra of matrix addition and multiplication :

- Multiplication is not commutative in general (AB usually does not equal BA , even if you're multiplying square matrices so that at least the product matrices are the same size).

But other properties you're used to do hold:

- $+$ is commutative $A + B = B + A$
- $+$ is associative $(A + B) + C = A + (B + C)$
- scalar multiplication distributes over $+$ $c(A + B) = cA + cB$.
- multiplication is associative $(AB)C = A(BC)$.
- matrix multiplication distributes over $+$ $A(B + C) = AB + AC$; $(A + B)C = AC + BC$

Exercise 5:

a) Verify these properties - except for the associative property for multiplication they're all easy to understand.

b) For the multiplicative associative property verify that at least the dimensions of the triple product matrices are the same.

c) Then check that for the matrices in exercises 3-4, it is indeed true that $(AC)B = A(CB)$.

Identity matrices: The $n \times n$ identity matrix $I_{n \times n}$ has one's down the diagonal (by which we mean the diagonal from the upper left to lower right corner), and zeroes elsewhere. For example,

$$I_{1 \times 1} = [1], \quad I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

In other words, $entry_{i\,i}(I_{n \times n}) = 1$ and $entry_{i\,j}(I_{n \times n}) = 0$ if $i \neq j$.

Exercise 6) Check that

$$A_{m \times n} I_{n \times n} = A, \qquad I_{m \times m} A_{m \times n} = A.$$

Hint: for the first equality show that the j^{th} columns of each side agree. For the second equality compare the i^{th} rows.
(That's why these matrices are called identity matrices - they are the matrix version of multiplicative identities, e.g. like multiplying by the number 1 in the real number system.)

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I.$$

In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark: A matrix A can have at most one inverse, because if

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so

$$B = C.$$

Exercise 7a) Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Inverse matrices are very useful in solving algebra problems. For example

Theorem: If A^{-1} exists then the only solution to $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is $\underline{\mathbf{x}} = A^{-1}\underline{\boldsymbol{b}}$.

Exercise 7b) Use the theorem and A^{-1} in 7a), to write down the solution to the system

$$\begin{aligned} x + 2y &= 5 \\ 3x + 4y &= 6 \end{aligned}$$

Exercise 8) Use matrix algebra to verify why the Theorem is true. Notice that the correct formula is $\underline{\mathbf{x}} = A^{-1}\underline{\boldsymbol{b}}$ and not $\underline{\mathbf{x}} = \underline{\boldsymbol{b}}\,A^{-1}$ (this second product can't even be computed!).

But where did that formula for A^{-1} come from?

Answer: Consider A^{-1} as an unknown matrix, $A^{-1}=X$. We want

$$A\,X=I.$$

In the 2 × 2 case, we can break this matrix equation down by columns:

$$A\left[\,col_1(X)\,\middle|\,col_2(X)\,\right]=\left[\begin{array}{c|c}1&0\\0&1\end{array}\right].$$

We've solved several linear systems for different right hand sides but the same coefficient matrix before, so let's do it again: the first column of $X=A^{-1}$ should solve $A\,\underline{x}=\begin{bmatrix}1\\0\end{bmatrix}$ and the second column should solve $A\,\underline{x}=\begin{bmatrix}0\\1\end{bmatrix}$.

Exercise 9: Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c}1&2&1&0\\3&4&0&1\end{array}\right]$$

to find the two columns of A^{-1} for the previous example.

Exercise 10: Will this always work? Can you find A^{-1} for

$$A:=\begin{bmatrix}1&5&1\\2&5&0\\2&7&1\end{bmatrix}?$$

(to be continued...)