Math 2250-1
Mon Sept 17
3.2-3.3 Gaussian elimination to solve linear systems of equations.

• There is a matlab introduction session today after class in LCB 115, 9:40-10:30. I hope most of you have finished the second computer project already, which is due tonight at 11:59 p.m. But if not, there is still time.

• On Friday we found the explicit solutions to two linear algebraic systems of 3 equations in 3 unknowns, using Gaussian elimination. In the first case there was one solution vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}.$$

In the second case one of the equations was redundant, and there was an entire line of solutions,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 + 3t \\ 4 - 2t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

Since this is a new week, let's start with fresh notes that incorporate the left-over material from last week. If you understood the systematic way we solved linear systems on Friday, great! If not, please review your own notes or the filled in details that I've posted on our lecture page, for how we came up with the explicit solutions in the two exercises above.

Exercise 1a) (same as 5b Wed) Find all solutions to the system below, using the augmented matrix and systematic elementary row operations.

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 8z = 23.$$

1b) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?

Summary: We write the linear system (LS) of m equations for the vector $\underline{x} = \left[x_1, x_2, \ldots x_n\right]^T$ of the n unknowns as

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n = b_1$$
$$a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{2n}\, x_n = b_2$$
$$\vdots \qquad\qquad \vdots \quad \vdots$$
$$a_{m1}\, x_1 + a_{m2}\, x_2 + \ldots + a_{mn}\, x_n = b_m$$

The matrix that we get by adjoining (augmenting) the right-side $\underline{b}$-vector to the coefficient matrix $A = \left[a_{ij}\right]$ is called the _augmented matrix_ $\langle A | \underline{b} \rangle$:

$$\left[\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\
\vdots & & & & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m
\end{array}\right]$$

Our goal is to find all the solution vectors $\underline{x}$ to the system - i.e. the _solution set_ .

Recall that the three types of _elementary equation operations_ that don't change the solution set to the linear system are

- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

And that when working with the augmented matrix $\langle A | \underline{b} \rangle$ these correspond to the three types of _elementary row operations:_

- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row with its sum with a multiple of a different row.

Gaussian elimination: Use elementary row operations and work column by column (from left to right) and row by row (from top to bottom) to first get the augmented matrix for an equivalent system of equations which is in

row-echelon form:
   (1)   All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.
   (2)   The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it.
                 (At this stage you could "backsolve" to find all solutions.)

Next, continue but by working from bottom to top and from right to left instead, so that you end with an augmented matrix that is in

reduced row echelon form: (1),(2), together with
  (3)   Each leading non-zero row entry has value 1 . Such entries are called "leading 1's "
  (4)   Each column that has (a row's) leading 1 has 0's in all the other entries.

Finally, read off how to explicitly specify the solution set, by "backsolving" from the reduced row echelon form.

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. All mathematical software will have a command to find the reduced row echelon form.

Exercise 2 Find all solutions to the system of 3 linear equations in 5 unknowns
$$x_1 - 2 x_2 + 3 x_3 + 2 x_4 + x_5 = 10$$
$$2 x_1 - 4 x_2 + 8 x_3 + 3 x_4 + 10 x_5 = 7$$
$$3 x_1 - 6 x_2 + 10 x_3 + 6 x_4 + 5 x_5 = 27 .$$
Here's the augmented matrix:
$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & | & 10 \\ 2 & -4 & 8 & 3 & 10 & | & 7 \\ 3 & -6 & 10 & 6 & 5 & | & 27 \end{bmatrix}$$

2a) Find a row-echelon form and backsolve to explicitly find the solution set.
2b) Then continue to the reduced row echelon form, and notice that when you backsolve it's easier than from the row echelon form. (Hint: the solution set is a three-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

Solution: After you write your solution in vector form and use vector addition and scalar multiplication to rewrite it, you might get

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 7 \\ 0 \end{bmatrix} + q \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix}; \quad q, t \in \mathbb{R}
$$

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1, -2, 3, 2, 1,
                     2, -4, 8, 3, 10,
                     3, -6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  ⟨A|b⟩;      # the mathematical augmented matrix doesn't actually have
             # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(⟨A|b⟩);
```

$$
\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix}
$$

(1)

```
> LinearSolve(A, b);
        #  this command will actually write down the general solution, using
        # Maple's way of writing free parameters, which actually makes
        # some sense.  Generally when there  are free parameters involved,
        #  there will be equivalent ways to express the solution that may
        # look different.  But usually Maple's version will look like yours,
        # because it's using the same algorithm and choosing the free parameters
        # the same way too.

>
```

<u>Exercise 3</u>: Consider the following combustion equation. Methane, CH_4, is the primary component of natural gas. We've all heard about the dangers of carbon monoxide (CO) buildup if natural gas is burned without enough oxygen O_2 , and that's related to:

$$a\ CH_4 + b\ O_2 \rightarrow c\ CO_2 + d\ H_2O + e\ CO.$$

<u>3a)</u> Balance the equation to deduce that

$$a = c + e$$
$$4\ a = 2\ d$$
$$2\ b = 2\ c + d + e\ .$$

<u>3b)</u> Order the variables c, d, e, a, b and write this homogeneous linear system in synthetic form. (Since there are 5 variables and only 3 equations, I expect that I'll be able to solve for the outputs c, d, e in terms of the inputs a, b .) Verify that an augmented matrix for this system is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 2 & 1 & 1 & 0 & -2 & 0 \end{array}\right].$$

<u>3c)</u> Use this reduced row echelon computation (notice that I didn't bother putting in that zero column on the end, because it stays zero for all the row operations), to express c, d, e in terms of a, b.

```
> A := Matrix(3, 5, [1, 0, 1, -1, 0, 0, 1, 0, -2, 0, 2, 1, 1, 0, -2]);
  ReducedRowEchelonForm(A);
```

$$A := \left[\begin{array}{ccccc} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 2 & 1 & 1 & 0 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -4 & 2 \end{array}\right] \hspace{2cm} \textbf{(2)}$$

>

<u>3d)</u> Discuss the implications for the resulting balanced equation

$$a\ CH_4 + b\ O_2 \rightarrow (-3\ a + 2\ b)\ CO_2 + 2\ a\ H_2O + (4\ a - 2\ b)\ CO$$