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Math 2250-1
Quiz 13 Solutions
December 7, 2012

Here is a system of two differential equations, which could be modeling two competing species:

$$x'(t) = 12x - 3x^2 - 2xy$$
$$y'(t) = 9y - y^2 - 3xy .$$

1) Find all equilibrium solutions for this system of differential equations, algebraically.

(4 points)

$$E_1: \quad 0 = 12x - 3x^2 - 2xy = x(12 - 3x - 2y)$$
$$E_2: \quad 0 = 9y - y^2 - 3xy = y(9 - y - 3x) .$$

Equilibrium points will make both equations true. E_1 is true if and only if $x = 0$ or $12 - 3x - 2y = 0$. E_2 is true if and only if $y = 0$ or $9 - y - 3x = 0$. So the four possibilities are

a) $x = 0, y = 0 \Rightarrow (x_*, y_*) = (0, 0)$
b) $x = 0, 9 - y - 3x = 0 \Rightarrow y = 9 \Rightarrow (x_*, y_*) = (0, 9)$
c) $12 - 3x - 2y = 0, y = 0 \Rightarrow x = 4 \Rightarrow (x_*, y_*) = (4, 0)$
d) $12 - 3x - 2y = 0, 9 - y - 3x = 0 \Rightarrow$

$$3x + 2y = 12$$
$$3x + y = 9 .$$

Subtracting equations implies $y = 3 \Rightarrow x = 2 \Rightarrow (x_*, y_*) = (2, 3)$.

2) One of the equilbrium solutions to the system above is $(x_*, y_*) = (2, 3)$. Find the linearized system of differential equations at that equilibrium point, and use eigenvalues to classify the equilibrium point. The pplane picture below may help inspire your work. If you had more time, I would maybe have asked you to sketch what the solutions to the linearized system look like, based on the linearization, eigenvalues, and eigenvectors.

(6 points)

$$J(x, y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 12 - 6x - 2y & -2x \\ -3y & 9 - 2y - 3x \end{bmatrix}$$

$$J(2, 3) = \begin{bmatrix} -6 & -4 \\ -9 & -3 \end{bmatrix}$$

So the linearized system at $(x_*, y_*) = (2, 3)$ is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ -9 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} .$$

To classify the equilibrium point we compute the eigenvalues of the Jacobian, via the characteristic polynomial

$$|J - \lambda I| = \begin{bmatrix} -6 - \lambda & -4 \\ -9 & -3 - \lambda \end{bmatrix} = (\lambda + 6)(\lambda + 3) - 36 = \lambda^2 + 9\lambda - 18 .$$

The roots of the characteristic polynomial are

$$\lambda = \frac{-9 \pm \sqrt{81 + 72}}{2} .$$

Since the two eigenvalues are real, with opposite signs, this equilibrium solution is an (unstable) saddle point.