Math 2250-1
Fri Oct 5
4.4 Basis and dimension, and studying subspaces.

There are two ways that you have encountered subspaces:
1) As the span of a collection of vectors (Theorem 1 discussed in detail on Wednesday). This is an explicit way to describe the subspace, because you are "listing" all of the vectors in it.

2) As the solution space to a homogenous matrix equation (discussed by Kyle in the problem session yesterday). This is an implicit way to describe the subspace because you're only saying the matrix equation that the vectors must satisfy, but you're not saying what the vectors are.

Here's what Kyle discussed:

Theorem 2: Let $A_{m \times n}$ be a matrix. Consider the solution space W of all solutions to the homogeneous matrix equation
$$A \underline{x} = \underline{0} \ ,$$
i.e.
$$W = \left\{ \underline{x} \in \mathbb{R}^n \ s.t. \ A_{m \times n} \underline{x} = \underline{0} \right\} .$$
Then $W \subseteq \mathbb{R}^n$ is a subspace. Furthermore, you can find a basis for this subspace by backsolving from the reduced row echelon form of A: once you write the solution in linear combination form
$$\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \ ... + c_k \underline{v}_k$$
the vectors $\underline{v}_1, \underline{v}_2, \ ... \ \underline{v}_k$ will always be linearly independent. Thus, since these vectors span W by construction, $\underline{v}_1, \underline{v}_2, \ ... \ \underline{v}_k$ are a basis for W.

Exercise 1) Check that (α), (β) hold for the solution space W to $A \underline{x} = \underline{0}$, i.e to verify that
$$W = \left\{ \underline{x} \in \mathbb{R}^n \ s.t. \ A_{m \times n} \underline{x} = \underline{0} \right\}$$
is indeed a subspace.

Verification:
α) Let $\underline{x} \in W, \underline{y} \in W$. We need to show that $\underline{x} + \underline{y} \in W$. In other words, is $A(\underline{x} + \underline{y}) = 0$?
Because $\underline{x} \in W, \underline{y} \in W$ we know $A \underline{x} = \underline{0}, A \underline{y} = \underline{0}$. Thus
$$A(\underline{x} + \underline{y}) = A \underline{x} + A \underline{y} = \underline{0} + \underline{0} = \underline{0} .$$
β) Let $\underline{x} \in W, c \in \mathbb{R}$. We need to show $c \underline{x} \in W$. In other words, is $A(c \underline{x}) = \underline{0}$? But
$$A(c \underline{x}) = c A \underline{x} = c \underline{0} = \underline{0} .$$
Thus W is a subspace.

Then Kyle went through the algorithm to find an explicit way to represent the solution space, which we actually already know from Chapter 3....see the next page.

Consider the matrix equation $A \underline{x} = \underline{0}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Exercise 2a)</u> Find a basis for the solution space $W = \{\underline{x} \in \mathbb{R}^6 \text{ s.t. } A\underline{x} = \underline{0}\}$ by backsolving, writing your explicit solutions in linear combination form, and extracting a basis. Explain why these vectors span the solution space and verify that they're linearly independent.

Solution: backsolve, realizing that the augmented matrices have final columns of zero. Label the free variables with the letter "t", and subscripts to show the non-leading 1 columns from which they arose:

$$x_6 = t_6, x_5 = t_5, x_4 = t_4, x_3 = -2\,t_4 + t_5 - 3\,t_6$$
$$x_2 = t_2, x_1 = -2\,t_2 - t_4 - t_5 - 2\,t_6.$$

In vector form and then linear combination form this is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix} = t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the four vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for the solution space:
- *They span the solution space by construction.*
- *They are linearly independent because if we set a linear combination equal to zero:*

$$t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix}$$

then looking in the second entry implies $t_2 = 0$, the fourth entry implies $t_4 = 0$, and similarly $t_5 = t_6 = 0$.

<u>2b)</u> What is the dimension of W ? How is the dimension related to the shape of the reduced row echelon form of A?

Because there are four vectors in a basis for this solution space, the dimension of the solution space is the number 4. Notice that this number is the number of free parameters, which is the number of columns of the reduced row echelon form that do not have leading ones.

In <u>Exercise 2,</u> we began by describing the subspace W implicitly, and then found a basis with which to express W explicitly as the span of that collection of basis vectors.

It's interesting that the same matrix $A_{m \times n}$ can occur for two different subspace problems, and this happened in your homework. That can be confusing.

(1) It could be you're trying to find a basis for the span of a collection of n vectors in $\mathbb{R}^m$. You make these columns these vectors are the columns of a matrix $A_{m \times n}$, compute the reduced row echelon form, and read off column dependencies in the original matrix. Then use these dependencies to delete dependent vectors from your collection until you're left with a basis for the span of the original collection.

(2) It could be you're trying to find a basis for the solution space in $\mathbb{R}^n$ to $A_{m \times n} \underline{x} = \underline{0}$, by reducing A and backsolving, as we just illustrated in <u>Exercise 2</u>.

We can illstrate (1) with the matrix we just used for (2):

<u>Exercise 3a)</u> Consider the six vectors in $\mathbb{R}^4$ that were the columns of the previous matrix A. Now find a basis for the subspace W spanned of these six vectors. Here's the same computation we used before:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

<u>3b)</u> What is the dimension of this subspace of $\mathbb{R}^4$?

Exercise 4)   There's a connection between the basis of the subspace of $\mathbb{R}^6$ in Exercise 2 and how we found the dependencies in Exercise 3, for the column vectors in $\mathbb{R}^4$. The connection is that, as we've talked about before, homogeneous solutions $\underline{c}$ to $A\,\underline{c} = \underline{0}$ correspond to column dependencies for A . Check this correspondence, by looking once more at the basis vectors for the solution space to $A\,\underline{c} = 0$ that you found in Exercise 2 and the column dependencies you found in 3a. Here were the Exercise 2 basis vectors and the reduced row echelon form computation (once again):

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark) Why each matrix only has one possible Reduced Row Echelon Form: The four conditions for rref (zero rows at bottom, non-zero rows start with leading ones, leading ones in successive rows move to the right, all column entries in any column containing a leading "1" are zero except for that "1"), together with specified column dependencies as above uniquely determine the rref matrix! You can see this with the matrix A above.

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic:

key fact: If n vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ span a subspace W then any collection $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text. Notice that it fits our intuition based on what we know in the special case that $W = \mathbb{R}^n$.)

Thus:
1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W. (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) If the dimension of W is the number n, and if $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete at least one of them and still have a spanning set, which would violate (1).)

4) If the dimension of W is the number n, and if $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$ are in W and are linearly independent, then they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector w_{n+1} not in their span and have a collection of $n+1$ still independent* vectors in W, violating the key fact.)

* Check: If $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$ are linearly independent, and $\underline{w}_{n+1} \notin span\{\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n\}$, then $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n, \underline{w}_{n+1}$ are also linearly independent: Let
$$c_1\underline{w}_1 + c_2\underline{w}_2 + \ldots + c_n\underline{w}_n + d\,\underline{w}_{n+1} = \underline{0}.$$
Since $\underline{w} \notin span\{\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{w}_{n+1}$ as a linear combination of $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$). But once $d = 0$, we have
$$c_1\underline{w}_1 + c_2\underline{w}_2 + \ldots + c_n\underline{w}_n = \underline{0}$$
which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$.