Math 2250-1
Tues Oct 23

5.4 Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations. We are focusing on this homogeneous differential equation for functions $x(t)$, which can arise in a variety of physics situations and that we derived for a mass-spring configuration:

$$m\,x'' + c\,x' + k\,x = 0\,.$$

• Finish discussing the "simple harmonic motion" that occurs in case the damping coefficient $c = 0$, using Monday's notes.

• Then discuss the possibilities that arise when the damping coefficient $c > 0$. There are three of them:

Case 2: damping

$$m\,x'' + c\,x' + k\,x = 0$$

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$$

rewrite as

$$x'' + 2\,p\,x' + \omega_0^2\,x = 0.$$

$\left(p = \frac{c}{2\,m}, \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2\,p\,r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}\,.$$

2a) ($p^2 > \omega_0^2$, or $c^2 > 4\,m\,k$). overdamped. In this case we have two negative real roots

$$r_1 < r_2 < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2\,e^{r_2 t} = e^{r_2 t}\left(c_1 e^{(r_1 - r_2)\,t} + c_2\right).$$

• solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

2b) ($p^2 = \omega_0^2$, or $c^2 = 4\,m\,k$) critically damped. Double real root $r_1 = r_2 = -p = -\dfrac{c}{2\,m}$.

$$x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right) .$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

2c) ($p^2 < \omega_0^2$, or $c^2 < 4\,m\,k$) underdamped. Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$$x(t) = e^{-p\,t}\left(A\cos\left(\omega_1 t\right) + B\sin\left(\omega_1 t\right)\right) = e^{-p\,t} C\cos\left(\omega_1 t - \alpha_1\right) .$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1 .

Exercise 1) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:

1a)

$$x'' + 6\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

```
> with(DEtools) :
> dsolve({x''(t) + 6·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
>
```

1b)

$$x'' + 10\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

```
> dsolve({x''(t) + 10·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
>
```

1c)

$$x'' + 2\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

```
> dsolve({x''(t) + 2·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
>
```

```
> with(plots):
> plot0 := plot(cos(3·t) + 1/2·sin(3·t), t = 0..4, color = red):
  plot1a := plot(exp(-3·t)·(1 + 9/2·t), t = 0..4, color = green):
  plot1b := plot(21/16·exp(-t) - 5/16·exp(-9·t), t = 0..4, color = blue):
  plot1c := plot(5/8·sqrt(2) e^(-t)·sin(2 sqrt(2)·t) + e^(-t)·cos(2 sqrt(2)·t), t = 0..4, color = black):
  display({plot0, plot1a, plot1b, plot1c}, title = `IVP with all damping possibilities`);
```

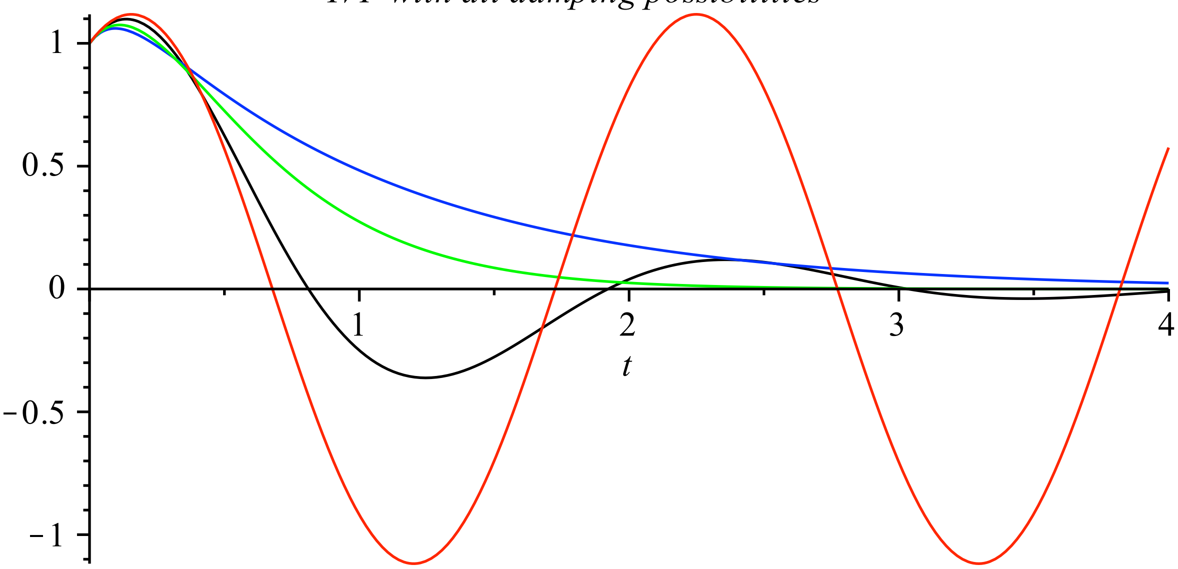
$$plot0 := plot\left(\cos(3\cdot t) + \frac{1}{2}\cdot\sin(3\cdot t),\ t = 0..4,\ color = red\right):$$

$$plot1a := plot\left(\exp(-3\cdot t)\cdot\left(1 + \frac{9}{2}\cdot t\right),\ t = 0..4,\ color = green\right):$$

$$plot1b := plot\left(\frac{21}{16}\cdot\exp(-t) - \frac{5}{16}\cdot\exp(-9\cdot t),\ t = 0..4,\ color = blue\right):$$

$$plot1c := plot\left(\frac{5}{8}\cdot\sqrt{2}\,\mathrm{e}^{-t}\cdot\sin\left(2\sqrt{2}\cdot t\right) + \mathrm{e}^{-t}\cdot\cos\left(2\sqrt{2}\cdot t\right),\ t = 0..4,\ color = black\right):$$

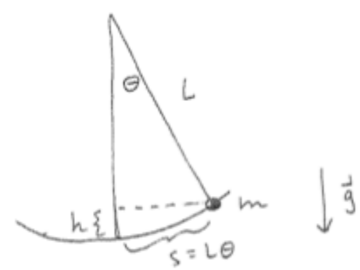
IVP with all damping possibilities



>

Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in our horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Tomorrow we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \dfrac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



$\theta = \theta(t)$

conservative system $KE + PE = \text{const.}$

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2}m L^2(\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$mL\theta'(L\theta'' + g\sin\theta) \equiv 0$

$mL\theta'$ $\neq 0$ except at isolated times

$\rightsquigarrow$ deduce eqtn of motion is $\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

linearize:

$\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

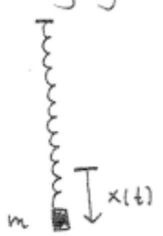
$\theta(t) = C\cos(\omega_0 t - \alpha)$

non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \dots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

2) hanging mass-spring:



$mx'' = -kx$

$mx'' + kx = 0$

$\boxed{x'' + \frac{k}{m}x = 0}$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?