Math 2250-1
Mon Oct 22

- Finish Friday's notes about complex exponentials and the algorithm for finding solution space bases to

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when

$$p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0$$

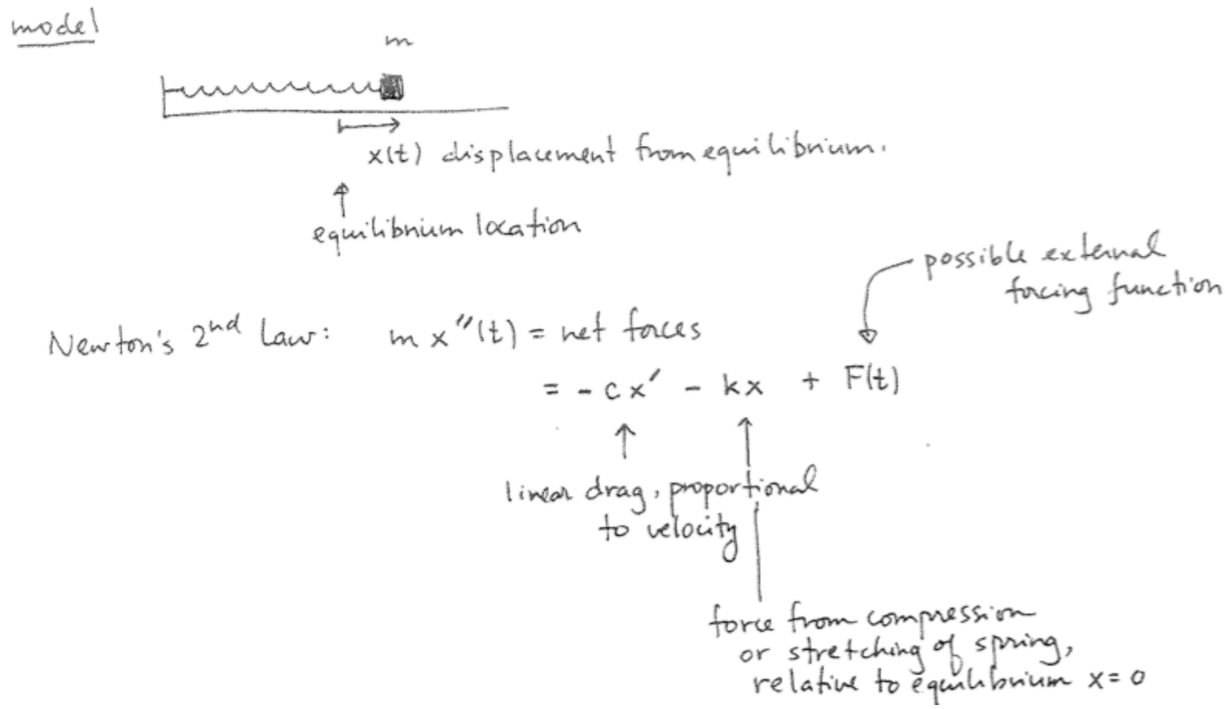
has complex roots.

Then begin applications...

5.4 Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$:

$$m\,x'' + c\,x' + k\,x = 0\,.$$



In section 5.4 we assume the external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero. Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t)$, $x'(t)$ are sufficiently small. k is called the Hooke's constant, and c is called the damping coefficient.

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{r\,t}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{r\,t}\left(m\,r^2 + c\,r + k\right) = e^{r\,t}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ pocesses...

Case 1) no damping ($c = 0$). (We will consider the cases with damping $c > 0$ tomorrow.)

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}\,x = 0\,.$$

$$p(r) = r^2 + \frac{k}{m}\,,$$

has roots

$$r^2 = -\frac{k}{m} \qquad \text{i.e.} \quad r = \pm\, i\sqrt{\frac{k}{m}}\ .$$

So the general solution is

$$x(t) = c_1\cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the <u>natural angular frequency</u>. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\,x = 0$$

is

$$x(t) = A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right).$$

This motion is called <u>simple harmonic motion.</u> The reason for this is that $x(t)$ can be rewritten as

$$x(t) = C\cos\left(\omega_0\,t - \alpha\right) = C\cos\left(\omega_0\left(t - \delta\right)\right)$$

in terms of an <u>amplitude</u> $C > 0$ and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ).

To see why functions of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

are equal (for appropriate choices of constants) to ones of the form

$$x(t) = C\cos(\omega_0 t - \alpha)$$

we use the very important the addition angle trigonometry identities, in this case the addition angle for *cosine*: Consider the possible equality

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) .$$

Exercise 1) Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two functions above are equal provided

(i)

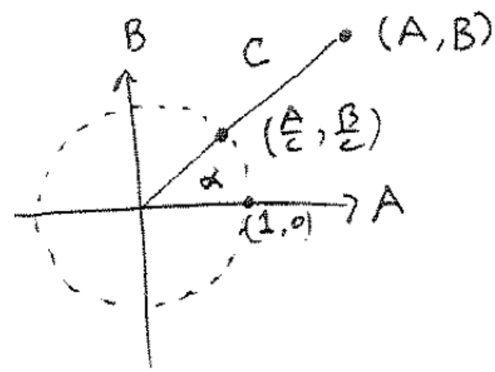
$$A = C\cos\alpha$$
$$B = C\sin\alpha$$

in case C, α are given, or

(ii)

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

in case A, B are given. These correspondences are best remembered using a diagram in the $A - B$ plane:



It is important to understand the behavior of the functions

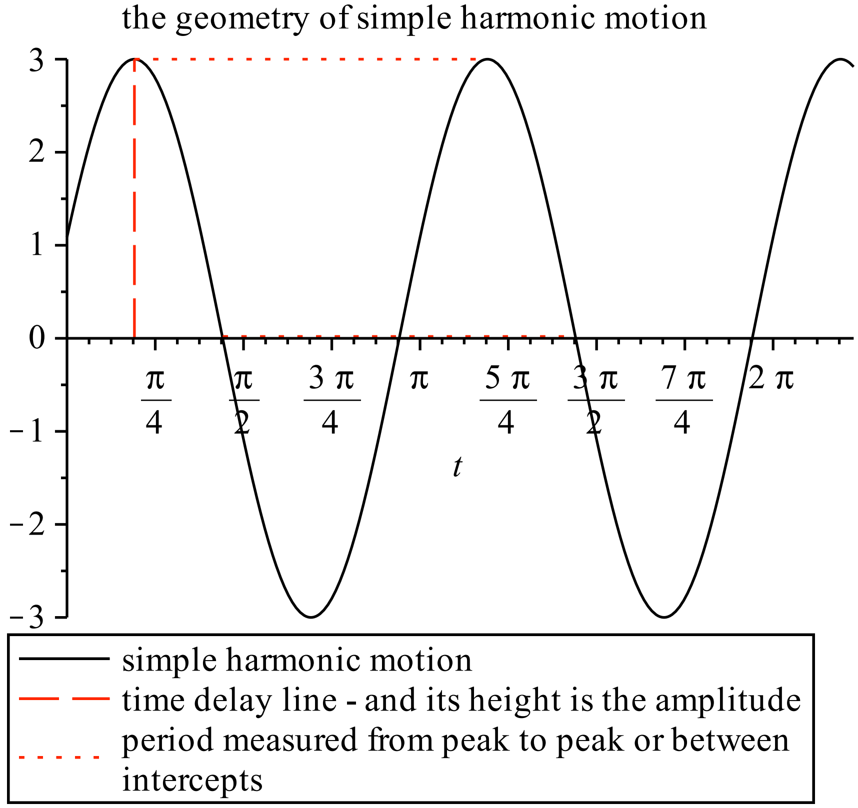
$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0(t-\delta))$$

and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} = \text{time/cycle.}$$



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0..7, color = black) :
  plot2 := plot([.6, t, t = 0..3.], linestyle = dash) :
  plot3 := plot(3, t = .6..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

Exercise 2) A mass of 2 *kg* oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 *m* from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

2a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$x'' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

2b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the commands below.

```
> unassign(`x`);
> with(plots):
  with(DEtools):
> dsolve({x''(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
> plot(rhs(%), t = 0..5, color = green);
>
```