Math 2250-1
Tues Oct 2
4.1 - 4.3 Concepts related to linear combinations of vectors.

Exercise 1) Vocabulary review:

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

The span of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent iff

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent iff

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are called a basis for the space they span iff

- Keep recalling that for vectors in $\mathbb{R}^m$ all linear combination questions can be reduced to matrix questions because any linear combination like the one on the left is actually just the matrix product on the right:

$$c_1\begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ : \\ a_{m2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ 0 \\ a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 a_{11} + c_2 a_{12} + \ldots \\ c_1 a_{21} + c_2 a_{22} + \ldots \\ : \\ c_1 a_{m1} + c_2 a_{m2} + \ldots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix}$$

- Now finish Exercises 5,6,7 from Monday's notes.

New vocabulary for today:

Definition: A vector space V is a collection of objects (called "vectors"), together with two operations: addition "+" and scalar multiplication "·" so that the following axioms hold:

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:
(a) $f + g = g + f$ (commutative property)
(b) $f + (g + h) = (f + g) + h$ (associative property)
(c) $\exists\, 0 \in V$ so that $f + 0 = f$ is always true. (additive identity)
(d) $\forall f \in V \; \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)
(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others).

Remark: If you look at those properties, they are exactly the ones we've been using in order to rewrite and simplify linear combinations of vectors in $\mathbb{R}^m$. If you look at them again, you'll see that they also hold for the addition and scalar multiplication of functions.

Definition: A subspace W of a vector space V is a (special) subset of V. In order to be called a subspace, this subset must be closed under addition and scalar multiplication, i.e. properties α, β above hold. As a result, W is itself a vector space, because it follows from α, β and the fact that W is a part of V that properties (a)-(h) also hold for all elements of W and all scalars in $\mathbb{R}$.

Exercise 2) The span of any collection of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ in $\mathbb{R}^m$ (or actually the span of any $f_1, f_2, \ldots f_n$ in some vector space V) is closed under addition and scalar multiplication, and so is a subspace of $\mathbb{R}^m$ (or V). Show this.

Exercise 3) Expressing a subspace as the span of a collection of vectors (as in Exercise 2) is an explicit way to specify all of the elements of the subspace. Another way that subspaces often get expressed is implicitly as the solution set to a homogeneous linear system of equations: Show that if $A_{m \times n}$ is a matrix, then the solution set to

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

is necessarily a subspace of $\mathbb{R}^n$. In other words, show that if you add any two solution vectors, you get a solution vector (α), and if you scalar multiply any solution vector, you get a solution vector (β).

Exercise 4) In the Monday Exercises 5-7, we were considering what we now realize from the general principles of Exercise 2 is a subspace, namely

$$W = span\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} \right\}.$$

4a) We also expressed W as the solution set to a very small linear homogeneous system of equations, as described in Exercise 3. How?

4b) If this plane W had been given to us implicitly as the solution set to the equation in your answer to 4a), how could we have found a basis for the plane directly from that equation? (Hint: backsolve, and write your explicit solutions in vector form.)

Exercise 5) Most subsets of a vector space V are actually not subspaces. Show that

5a) $W = \left\{ [x, y]^T \in \mathbb{R}^2 \ s.t.\ x^2 + y^2 = 1 \right\}$ is not a subspace of $\mathbb{R}^2$.

5b) $W = \left\{ [x, y]^T \in \mathbb{R}^2 \ s.t.\ y = 3\,x + 1 \right\}$ is not a subspace of $\mathbb{R}^2$.

5c) $W = \left\{ [x, y]^T \in \mathbb{R}^2 \ s.t.\ y = 3\,x \right\}$ is a subspace of $\mathbb{R}^2$. Then find a basis for this subspace.

Exercise 6) Can you use matrix theory to show that the only subspaces of $\mathbb{R}^2$ are
(0) The single vector $[0, 0]^T$, or
(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or
(2) All of $\mathbb{R}^2$.
Exercise 7) Can you use matrix theory to show that the only subspaces of $\mathbb{R}^3$ are
(0) The single vector $[0, 0, 0]^T$, or
(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or
(2) A plane through the origin, i.e.$span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ where $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$ are linearly independent, or
(3) All of $\mathbb{R}^3$.
Exercise 8) The number of vectors in a basis for a subspace is called the dimension of the subspace. What are the dimensions of the subspaces in Exercise 6 and Exercise 7?