Math 2250-1
Tues Oct 16
5.1-5.2 n^{th} order linear differential equations, and vector space theory connections.

Recap: Yesterday we started talking about <u>second order linear differential equations</u> for $y = y(x)$:
$$y'' + p(x)y' + q(x)y = f(x).$$
The differential equation is called <u>homogeneous.</u> in case $f(x)$ is the zero function, i.e. in case
$$y'' + p(x)y' + q(x)y = 0 .$$
Otherwise its called <u>inhomogeneous.</u>

• For the "operator" $L(y) := y'' + p(x)y' + q(x)y$ we checked the <u>linearity properties</u>

  (1) $L(y_1 + y_2) = L(y_1) + L(y_2)$
  (2) $L(cy) = cL(y), c \in \mathbb{R} .$

As a result of (just) these linearity properties we were able to understand the following two facts:

Theorem 0: The solution space to the homogeneous linear DE
$$y'' + p(x)y' + q(x)y = 0$$
is a subspace.

Theorem 1: The general solution to the <u>inhomogeneous</u> second order linear DE
$$y'' + p(x)y' + q(x)y = f(x)$$
is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

At the end of class we looked at the existence-uniqueness theorem for initial value problems:

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the <u>initial value problem</u>
$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

• Finish Monday's notes and examples to illsturate these facts, as well as the following result about the dimension of the solution space to the homogeneous DE:

Theorem 3: The solution space to the second order homogeneous linear differential equation
$$y'' + p(x)y' + q(x)y = 0$$
is 2-dimensional.

After carefully finishing Monday's notes it's not too hard to see how the discussion about 2nd order DE's generalizes to n^{th} order DE's. We can read this portion of the notes together and see that the ideas correspond exactly to our discussion for $n = 2$.

<u>Definition:</u> An nth order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form
$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = F(x).$$
We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x)$ $\neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$
($a_{n-1}, \ldots a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I.)
This DE is called <u>linear</u> because the operator L defined by
$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y$$
satisfies the so-called <u>linearity properties</u>
  (1) $L(y_1 + y_2) = L(y_1) + L(y_2)$
  (2) $L(c\,y) = c\,L(y)$, $c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when $n = 2$, that we checked yesterday. Then, since the $y = y_P + y_H$ proof only depended on the linearity properties of L, just like yesterday, we deduce both of Theorems 0 and 1:*

Theorem 0: The solution space to the homogeneous linear DE
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$
is a subspace.

Theorem 1: The general solution to the <u>nonhomogeneous</u> n^{th} order linear DE
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$
is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Later in the course we'll understand n^{th} order existence uniqueness theorems for initial value problems, in a way analogous to how we understood the first order theorem using slope fields, but let's postpone that discussion and just record the following true theorem as a fact:

Theorem 2 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x),... a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the <u>initial value problem</u>

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = f$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$y''(x_0) = b_2$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

Just as for the case $n = 2$, the existence-uniqueness theorem lets you figure out the dimension of the solution space to homogeneous linear differential equations. The proof is conceptually the same, but messier to write down because the vectors and matrices are bigger.

Theorem 3: The solution space to the nth order homogeneous linear differential equation
$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y \equiv 0$$
is n-dimensional.

<u>proof:</u> By the existence half of <u>Theorem 2</u>, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), ... y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, ..., \begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

<u>Claim:</u> In this case, the solutions $y_1, y_2, ... y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$
\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.
$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the <u>Wronskian matrix</u> at x_0 times the vector of linear combination coefficients:

$$
\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.
$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$
\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}
$$

has a unique solution $\underline{c}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$
y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.
$$

Thus $y_1, y_2, \ldots y_n$ <u>span</u> the solution space. If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then because the zero function has zero initial vector $[b_0, b_1, \ldots b_{n-1}]^T$ the matrix equation above implies that $[c_1, c_2, \ldots c_n]^T = \underline{0}$, so $y_1, y_2, \ldots y_n$ are also <u>linearly independent</u>. Thus, $y_1, y_2, \ldots y_n$ are a <u>basis</u> for the solution space and the general solution to the homogeneous DE can be written as

$$
y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.
$$

$\square$

Let's do some new exercises that tie these ideas together.

Exercise 1) Consider the 3rd order linear homogeneous DE for $y(x)$:
$$y''' + 3\,y'' - y' - 3\,y = 0.$$
Find a basis for the 3-dimensional solution space, and the general solution. Make sure to use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

Exercise 2a) Find the general solution to
$$y''' + 3\,y'' - y' - 3\,y = 6 .$$
Hint: First try to find a particular solution ... try a constant function.

b) Set up the linear system to solve the initial value problem for this DE, with
$y(0) = -1, y'(0) = 2, y''(0) = 7$.

for fun now, but maybe not just for fun later:
> $with(DEtools)$:
  $dsolve(\{y'''(x) + 3 \cdot y''(x) - y'(x) - 3 \cdot y(x) = 6, y(0) = -1, y'(0) = 2, y''(0) = 7\})$;