

Math 2250-1
Fri Nov 9

Today and Monday we'll discuss interesting applications of Laplace transforms to differential equations, using sections 10.4,10.5,EP7.6. The new table entries we'll focus on are

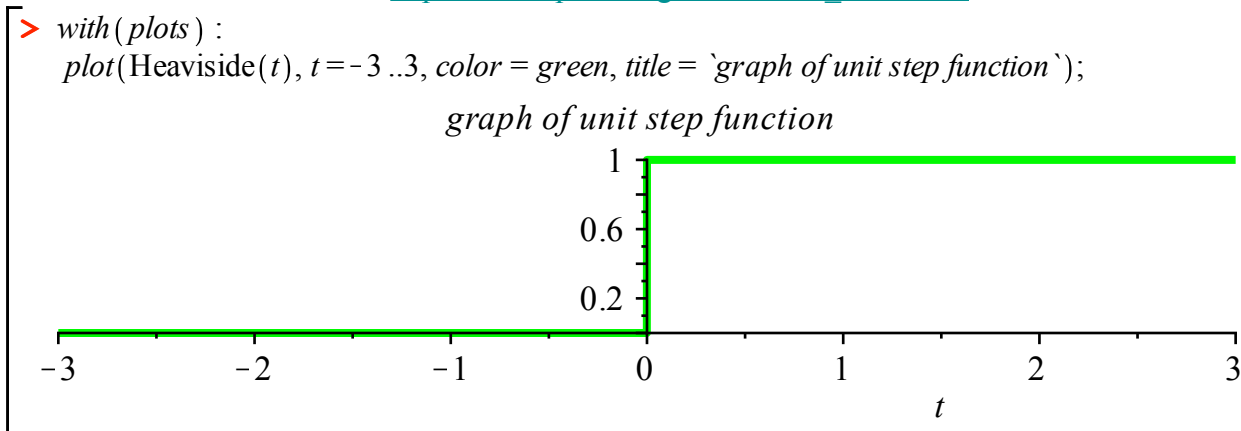
$f(t)$ with $ f(t) \leq Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st} dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-as}}{s}$	for turning components on and off at $t=a$.
$f(t-a)u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\delta(t-a)$	e^{-as}	unit impulse/delta "function"
$\int_0^t f(\tau)g(t-\tau) d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products
$f(t)$ with period p	$\frac{1}{1-e^{-ps}} \int_0^p f(t)e^{-st} dt$	for periodic forcing which is not sinusoidal.

The unit step function with jump at $t=0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}.$$

Its graph is shown below. Notice that this function is called the "Heaviside" function in Maple, after the person who popularized it (among a lot of other accomplishments) and not because it's heavy on one side.

http://en.wikipedia.org/wiki/Oliver_Heaviside

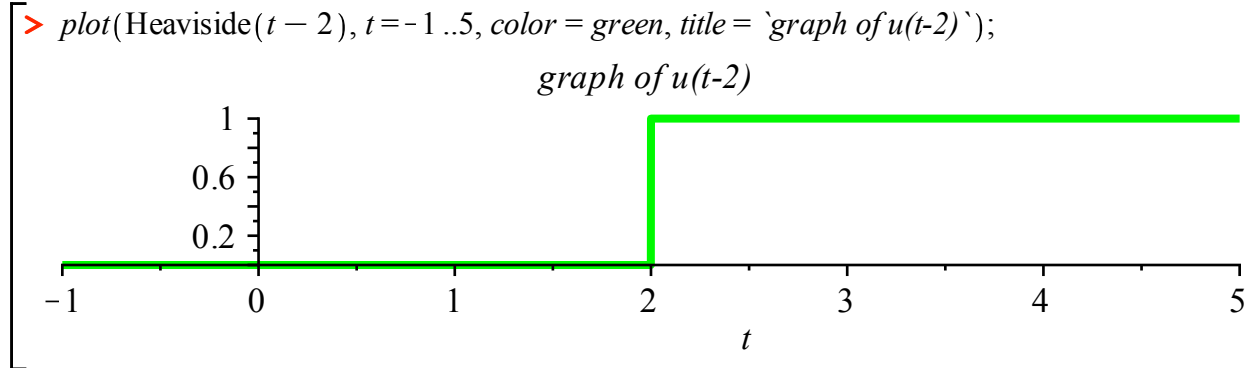


Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform it doesn't actually matter what we define $u(0)$ to be.

Then

$$u(t - a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

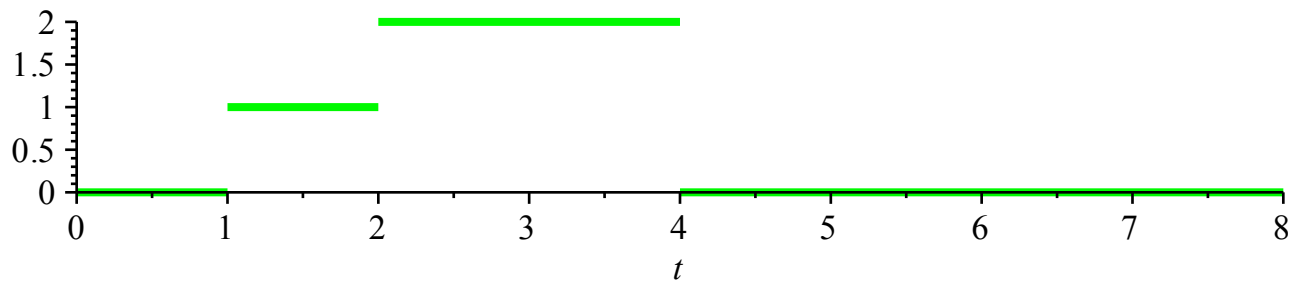
and has graph that is a horizontal translation of the original graph, e.g. for $a = 2$:



Exercise 1) Verify the table entries

$u(t - a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t - a) u(t - a)$	$e^{-a s} F(s)$	more complicated on/off

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from the previous assignment - although you found the Laplace transform of that step function directly from the definition.



```
[ > plot(Heaviside(t - 1) + Heaviside(t - 2) - 2·Heaviside(t - 4), t = 0 .. 8, color = green);
    with(inttrans) :
    laplace(Heaviside(t - 1) + Heaviside(t - 2) - 2·Heaviside(t - 4), t, s);
```

Setup: an under-employed mathematician/engineer/scientist
(your choice)
likes to take his/her child to the swings...

recall pendulum (linearized) eqn, without forcing, for $\theta = \theta(t)$

$$L\theta'' + g\theta = 0$$



$$\leadsto x'' + g \frac{x}{L} = 0$$

$$\leadsto mx'' + \frac{mg}{L}x = F_0 \cos \omega t \quad \leftarrow \text{parent forcing (?!)}$$

$$\begin{aligned} x(t) &= L \sin \theta(t) \\ &\approx L\theta \quad \text{for small } \theta \\ \text{so } x'' &\approx L\theta'' \end{aligned}$$

$$\begin{aligned} \leadsto x'' + \frac{g}{L}x &= \frac{F_0}{m} \cos \omega t \\ x'' + \omega_0^2 x &= \frac{F_0}{m} \cos \omega t \end{aligned}$$

parent pushes sinusoidally for
exactly 5 cycles, and
with $\frac{F_0}{m} = 0.2$ and then releases:

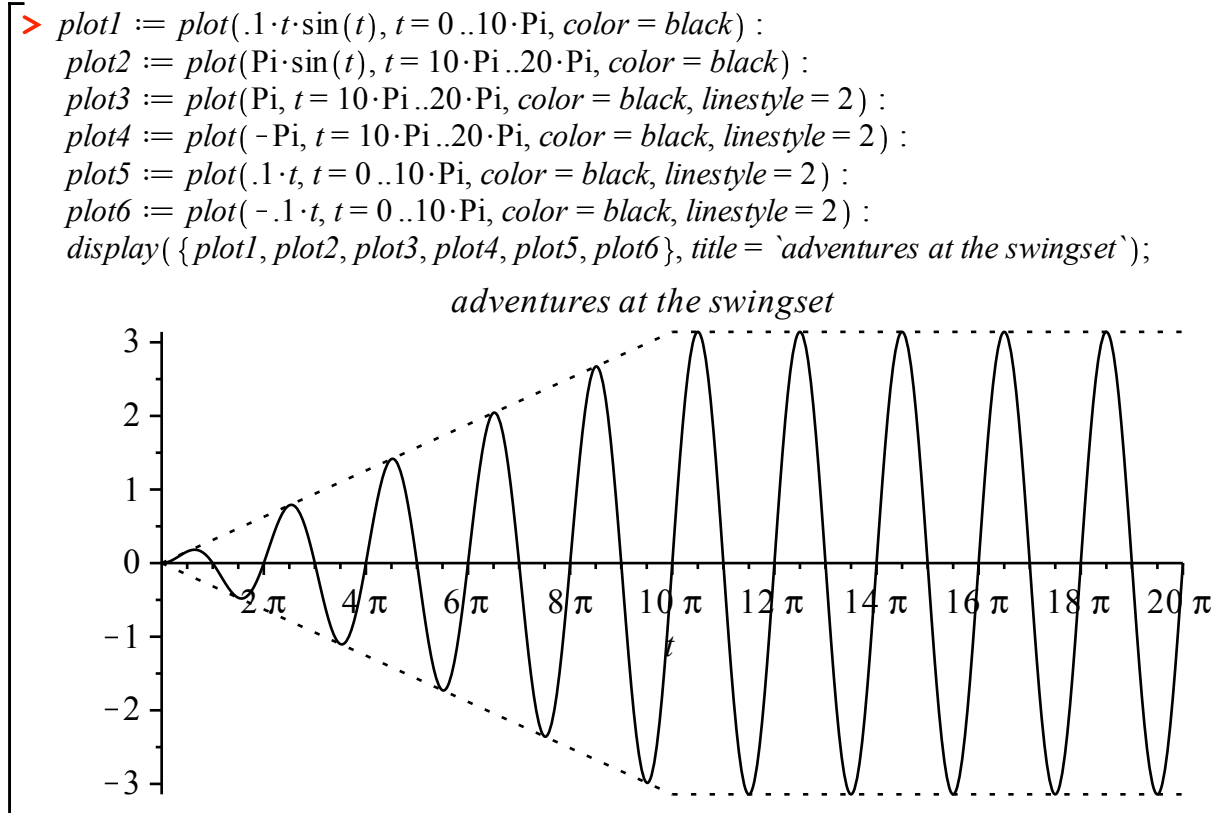
for resonance $\omega = \omega_0$
construct swing with $L = g \approx 9.8$ m.
so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds
:)

Exercise 3a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$\begin{aligned} x''(t) + x(t) &= 0.2 \cos(t) (1 - u(t - 10\pi)) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned}$$

3b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

Pictures for the swing:



Alternate approach via Chapter 5:

step 1) solve

$$\begin{aligned}
 x''(t) + x(t) &= .2 \cos(t) \\
 x(0) &= 0 \\
 x'(0) &= 0
 \end{aligned}$$

for $0 \leq t \leq 10\pi$.

step 2) Then solve

$$\begin{aligned}
 y''(t) + y(t) &= 0 \\
 y(0) &= x(10\pi) \\
 y'(0) &= x'(10\pi)
 \end{aligned}$$

and set $x(t) = y(t - 10)$ for $t > 10$.

Laplace transform convolution

For Laplace transforms the convolution of f and g is defined to be

$$f * g(t) := \int_0^t f(\tau) g(t - \tau) d\tau.$$

Exercise 4) Show $f * g(t) = g * f(t)$ by doing a change of variables in the convolution integral.

$\int_0^t f(\tau) g(t - \tau) d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products
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Exercise 5) Verify that the convolution integral table entry is correct, for

$$f(t) = \sin(t) \quad F(s) = \frac{1}{s^2 + 1}$$

$$g(t) = \cos(t) \quad G(s) = \frac{s}{s^2 + 1}$$

$$f * g(t) \quad F(s)G(s) = \frac{s}{(s^2 + 1)^2}.$$

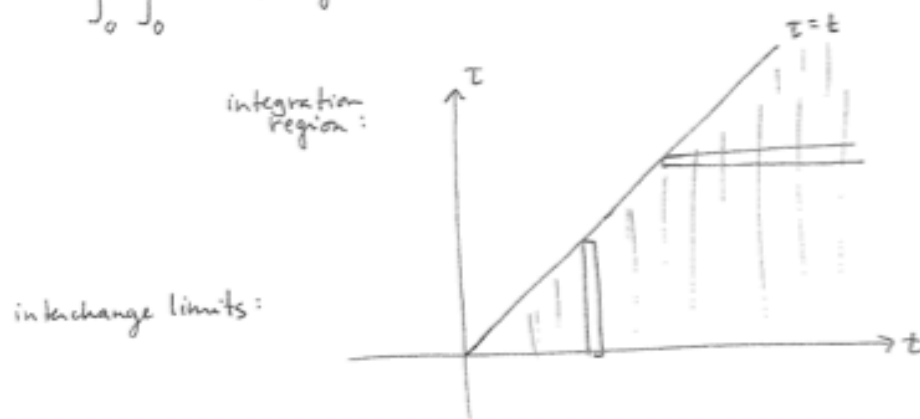
Hint: you might use the trig identity

$$\sin(\tau)^2 = \frac{1 - \cos(2\tau)}{2}.$$

proof of convolution theorem:

(is a good review of iterated integrals)

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^{\infty} e^{-st} \left(\int_0^t f(\tau) g(t-\tau) d\tau \right) dt \\ &= \int_0^{\infty} \int_0^t e^{-st} f(\tau) g(t-\tau) d\tau dt \end{aligned}$$



$$\begin{aligned} &= \int_0^{\infty} \int_{\tau}^{\infty} e^{-st} f(\tau) g(t-\tau) dt d\tau \\ &= \int_0^{\infty} \int_{\tau}^{\infty} e^{-s\tau} f(\tau) e^{-s(t-\tau)} g(t-\tau) dt d\tau \quad (\text{pattern recognition}) \end{aligned}$$

$$\begin{aligned} &= \int_0^{\infty} e^{-s\tau} f(\tau) \left[\int_{\tau}^{\infty} e^{-s(t-\tau)} g(t-\tau) dt \right] d\tau \\ &\quad \begin{array}{l} \tilde{t} = t - \tau \\ d\tilde{t} = dt \end{array} \\ &\quad \underbrace{\left[\int_0^{\infty} e^{-s\tilde{t}} g(\tilde{t}) d\tilde{t} \right]}_{G(s)} \end{aligned}$$

$$= G(s) \int_0^{\infty} e^{-s\tau} f(\tau) d\tau$$

$$= G(s) F(s) \quad !!$$

$f(t), \text{ with } f(t) \leq Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st} dt \text{ for } s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s} \quad (s > 0)$	☐
t	$\frac{1}{s^2}$	☐
t^2	$\frac{2}{s^3}$	☐
$t^n, n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s - \alpha} \quad (s > \Re(\alpha))$	☐
$\cos(kt)$	$\frac{s}{s^2 + k^2} \quad (s > 0)$	☐
$\sin(kt)$	$\frac{k}{s^2 + k^2} \quad (s > 0)$	☐
$\cosh(kt)$	$\frac{s}{s^2 - k^2} \quad (s > k)$	☐
$\sinh(kt)$	$\frac{k}{s^2 - k^2} \quad (s > k)$	☐
$e^{at}\cos(kt)$	$\frac{(s - a)}{(s - a)^2 + k^2} \quad (s > a)$	☐
$e^{at}\sin(kt)$	$\frac{k}{(s - a)^2 + k^2} \quad (s > a)$	☐
$e^{at}f(t)$	$F(s - a)$	☐
$u(t - a)$	$\frac{e^{-as}}{s}$	
$f(t - a) u(t - a)$	$e^{-as}F(s)$	
$\delta(t - a)$	e^{-as}	
$f'(t)$	$s F(s) - f(0)$	☐
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$	☐
$f^{(n)}(t), n \in \mathbb{N}$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$	☐
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$	☐
$t f(t)$	$-F'(s)$	☐
$t^2 f(t)$	$F''(s)$	☐

$t^n f(t), n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$(-1)^n F^{(n)}(s)$ $\int_s^\infty F(\sigma) d\sigma$	☐
$t \cos(kt)$ $\frac{1}{2k} t \sin(kt)$ $\frac{1}{2k^3} (\sin(kt) - kt \cos(kt))$ $t e^{at}$ $t^n e^{at}, n \in \mathbb{Z}$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$ $\frac{1}{(s - a)^2}$ $\frac{n!}{(s - a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau)g(t - \tau) d\tau$	$F(s)G(s)$	☐
$f(t)$ with period p	$\frac{1}{1 - e^{-ps}} \int_0^p f(t)e^{-st} dt$	☐

Laplace transform table