

Math 2250-1
Mon Nov 5

Homework problem session Tuesday 12:55-1:45 JTB 310, for HW due Wednesday.
Go over last spring exam Tuesday 4:30-6:00 Web L104 (here).
Today's office hours are 12:30-1:20 (rather than 2:20-3:00).

10.1-10.3 Laplace transform, and application to DE IVPs, including Chapter 5.
Today we'll continue to fill in the Laplace transform table (on page 2), and to use the table entries to solve linear differential equations.

Exercise 1) (to review) Use the table to compute

1a) $\mathcal{L}\{4 - 5 \cos(3t) + 2e^{-4t} \sin(12t)\}(s)$

1b) $\mathcal{L}^{-1}\left\{\frac{2}{s-2} + \frac{1}{s^2 + 2s + 5}\right\}(t) .$

Exercise 2) (to review) Use Laplace transforms to solve the IVP we didn't get to on Friday, for an underdamped, unforced oscillator DE:

$$\begin{aligned}x''(t) + 6x'(t) + 34x(t) &= 0 \\x(0) &= 3 \\x'(0) &= 1\end{aligned}$$

$f(t)$, with $ f(t) \leq C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t} dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	<input type="checkbox"/>
1 t t^2 $t^n, n \in \mathbb{N}$	$\frac{1}{s} \quad (s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	<input type="checkbox"/>
$e^{\alpha t}$	$\frac{1}{s - \alpha} \quad (s > \Re(\alpha))$	<input type="checkbox"/>
$\cos(k t)$	$\frac{s}{s^2 + k^2} \quad (s > 0)$	<input type="checkbox"/>
$\sin(k t)$	$\frac{k}{s^2 + k^2} \quad (s > 0)$	<input type="checkbox"/>
$\cosh(k t)$	$\frac{s}{s^2 - k^2} \quad (s > k)$	
$\sinh(k t)$	$\frac{k}{s^2 - k^2} \quad (s > k)$	
$e^{a t} \cos(k t)$	$\frac{(s - a)}{(s - a)^2 + k^2} \quad (s > a)$	<input type="checkbox"/>
$e^{a t} \sin(k t)$	$\frac{k}{(s - a)^2 + k^2} \quad (s > a)$	<input type="checkbox"/>
$f'(t)$ $f''(t)$ $f^{(n)}(t), n \in \mathbb{N}$ $\int_0^t f(\tau) d\tau$	$s F(s) - f(0)$ $s^2 F(s) - s f(0) - f'(0)$ $s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	
$t f(t)$ $t^2 f(t)$ $t^n f(t), n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$\frac{-F'(s)}{F''(s)}$ $(-1)^n F^{(n)}(s)$ $\int_s^\infty F(\sigma) d\sigma$	
$t \cos(k t)$ $\frac{1}{2 k} t \sin(k t)$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$	

$\frac{1}{2k^3}(\sin(kt) - kt \cos(kt))$ $t e^{at}$ $t^n e^{at}, n \in \mathbb{Z}$	$\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	
$e^{at} f(t)$ $u(t-a), a > 0$ $f(t-a) u(t-a)$	$F(s-a)$ $\frac{e^{-as}}{s}$ $e^{-as} F(s)$	
more later!		

Laplace transform table

work down the table ...

$$\underline{3a)} \quad \mathcal{L}\{\cosh(kt)\}(s) = \frac{s}{s^2 - k^2}$$

$$\underline{3b)} \quad \mathcal{L}\{\sinh(kt)\}(s) = \frac{k}{s^2 - k^2}.$$

Exercise 4) Derive $\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)$ and then use it to show

$$\underline{a)} \quad \mathcal{L}\{f''(t)\}(s) = s^2 F(s) - sf(0) - f'(0),$$

$$\underline{b)} \quad \mathcal{L}\{f'''(t)\}(s) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0),$$

$$\underline{c)} \quad \mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0), n \in \mathbb{N}.$$

$$\underline{d)} \quad \mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\}(s) = \frac{F(s)}{s}.$$

Exercise 5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + 4)}\right\}(t)$

- a) using the result of 4d.
b) using partial fractions.

Exercise 6) Show $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, n \in \mathbb{N}$, using the results of 4.

A harder table entry to understand - go through this computation and see why it seems reasonable, even though there's one step that we don't completely justify. The table entry is

$tf(t)$	$-F'(s)$
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Here's how we get it:

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st} dt$$

$$\Rightarrow \frac{d}{ds}F(s) = \frac{d}{ds} \int_0^\infty f(t)e^{-st} dt = \int_0^\infty \frac{d}{ds}f(t)e^{-st} dt.$$

It's this last step which is true, but needs more justification. We know that the derivative of a sum is the sum of the derivatives, and the integral is a limit of Riemann sums, so this step does at least seem reasonable. The rest is straightforward:

$$\int_0^\infty \frac{d}{ds}f(t)e^{-st} dt = \int_0^\infty f(t)(-t)e^{-st} dt = -\mathcal{L}\{tf(t)\}(s) \quad \square.$$

For resonance and other applications ...

Exercise 7) Use $\mathcal{L}\{tf(t)\}(s) = -F'(s)$ to show

a) $\mathcal{L}\{t \cos(kt)\}(s) = \frac{s^2 - k^2}{(s^2 + k^2)^2}$

b) $\mathcal{L}\left\{\frac{1}{2k} t \sin(kt)\right\}(s) = \frac{s}{(s^2 + k^2)^2}$

c) Then use a) and the identity

$$\frac{1}{(s^2 + k^2)^2} = \frac{1}{2k^2} \left(\frac{s^2 + k^2}{(s^2 + k^2)^2} - \frac{s^2 - k^2}{(s^2 + k^2)^2} \right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + k^2)^2}\right\}(t) = \frac{1}{2k^2} \left(\frac{1}{k} \sin(kt) - t \cos(kt) \right).$$

Notice how the Laplace transform table is set up to use partial fraction decompositions. And be amazed at how it lets you quickly deduce the solutions to important DE IVPs, like this resonance problem:

Exercise 8) Use Laplace transforms to write down the solution to

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega_0 t)$$

$$x(0) = x_0$$

$$x'(0) = v_0.$$