

Math 2250-1

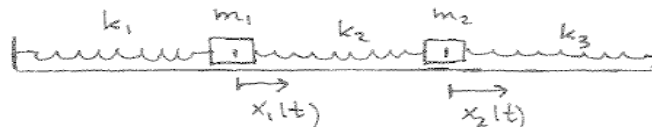
Tues Nov 27

7.4 Mass-spring systems.

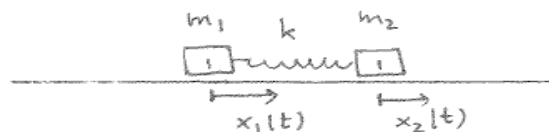
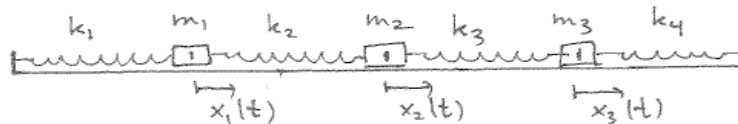
- Yesterday we considered conservative coupled mass spring systems with n masses, and discussed how to find bases for the $2n$ -dimensional solution spaces to the resulting system of n second order linear homogeneous differential equations

$$M \underline{x}''(t) = K \underline{x} \Rightarrow \underline{x}''(t) = A \underline{x}.$$

We focused on the case of 2 masses and three springs,



although in your homework you'll also consider configurations like these:



In the example case from yesterday where all three Hooke's constants are the same, and the two mass values are the same, we found a basis for the 4-dimensional solution space, where each basis function was either $\cos(\omega_j t) \underline{v}_j$ or $\sin(\omega_j t) \underline{v}_j$, where the (two) \underline{v}_j 's were eigenvectors of A , with negative eigenvalues $\lambda_j = -\omega_j^2$ ($\omega_j = \sqrt{-\lambda_j}$).

Exercise 1) Return to yesterday's notes, where we had just written down the solution at the top of page 4. Then complete exercise 3 there, verifying that you can solve each initial value problem for this second order system with a unique choice of the linear combination coefficients c_1, c_2, c_3, c_4 .

The two mass, three spring system....Experiment!

Data: Each mass is 50 grams. Each spring mass is 10 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and a mass of 50 grams stretches the spring 15.6 centimeters. (We should recheck this since it's last fall's data; we should also test the spring's "Hookiness"). With the old numbers we get Hooke's constant

$$\begin{aligned} &> \text{Digits} := 4 : \\ &> \text{solve}(k \cdot .156 = .05 \cdot 9.806, k) \end{aligned} \quad 3.143 \quad (1)$$

Here's Maple confirmation for some of our work yesterday:

$$\begin{aligned} &> \text{with(LinearAlgebra)} : \\ &A := \text{Matrix}\left(2, 2, \left[-\frac{2 \cdot k}{m}, \frac{k}{m}, \frac{k}{m}, -\frac{2 \cdot k}{m}\right]\right); \\ &\text{Eigenvectors}(A); \end{aligned} \quad A := \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} -\frac{3k}{m} \\ k \\ -\frac{k}{m} \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Predict the two natural periods from the model and our experimental value of k, m . Then make the system vibrate in each mode individually and compare your prediction to the actual periods of these two fundamental modes.

ANSWER: If you do the model correctly and my office data is close to our class data, you will come up with theoretical natural periods of close to .46 and .79 seconds. I predict that the actual natural periods are a little longer, especially for the slow mode. (In my office experiment I got periods of 0.482 and 0.855 seconds.) What happened?

EXPLANATION: The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 5.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator <http://www.math.utah.edu/~korevaar/2250fall12/oct24.pdf>, assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an A -matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6k}{6m + 5m_s}$$

$$\lambda_2 = -\frac{6k}{2m + m_s}.$$

(Hints: the "M" matrix is not diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

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> m := .05;
  ms := .010;
  k := 3.143;

  Omega1 := sqrt( (6*k) / (6*m + 5*ms) );
  Omega2 := sqrt( (6*k) / (2*m + ms) );
  T1 := evalf( (2*Pi) / Omega1 );
  T2 := evalf( (2*Pi) / Omega2 );

  m := 0.05
  ms := 0.010
  k := 3.143
  Omega1 := 7.340
  Omega2 := 13.09
  T1 := 0.8559
  T2 := 0.4801

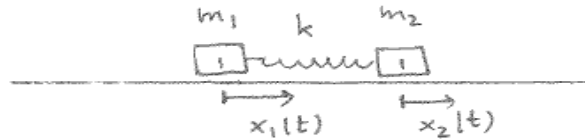
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(3)

of .856 and .480 seconds per cycle. Is that closer?

Challenge: If you can construct (and explain to me in my office) a correct derivation of the eigenvalues /eigenvectors I claim above, by a conservation of energy argument that takes the spring masses and kinetic energy into account, then you can replace your lowest homework score this term with "20". Due date: Next Friday December 7, 5:00 p.m.

Exercise 2) Consider a train with two cars connected by a spring:



2a) Derive the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero)

2b) Find the eigenvalues and eigenvectors. Then find the general solution. For $\lambda = 0$ and its corresponding eigenvector \underline{v} verify that you get two solutions

$$\underline{x}(t) = \underline{v} \text{ and } \underline{x}(t) = t \underline{v},$$

rather than the expected $\cos(\omega t) \underline{v}$, $\sin(\omega t) \underline{v}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems.

Forced oscillations (still undamped):

$$\begin{aligned} M \underline{\mathbf{x}}''(t) &= K \underline{\mathbf{x}} + \underline{\mathbf{F}}(t) \\ \Rightarrow \underline{\mathbf{x}}''(t) &= A \underline{\mathbf{x}} + M^{-1} \underline{\mathbf{F}}(t) . \end{aligned}$$

If the forcing is sinusoidal,

$$\begin{aligned} M \underline{\mathbf{x}}''(t) &= K \underline{\mathbf{x}} + \cos(\omega t) \underline{\mathbf{G}}_0 \\ \Rightarrow \underline{\mathbf{x}}''(t) &= A \underline{\mathbf{x}} + \cos(\omega t) \underline{\mathbf{F}}_0 \end{aligned}$$

with $\underline{\mathbf{F}}_0 = M^{-1} \underline{\mathbf{G}}_0$.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_p(t) + \underline{\mathbf{x}}_h(t) ,$$

and we've just discussed how to find the homogeneous solutions $\underline{\mathbf{x}}_h(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\mathbf{x}}_p(t) = \cos(\omega t) \underline{\mathbf{c}}$$

where the vector $\underline{\mathbf{c}}$ is what we need to find.

Exercise 3) Substitute the guess $\underline{\mathbf{x}}_p(t) = \cos(\omega t) \underline{\mathbf{c}}$ into the DE system

$$\underline{\mathbf{x}}''(t) = A \underline{\mathbf{x}} + \cos(\omega t) \underline{\mathbf{F}}_0$$

to find a matrix algebra formula for $\underline{\mathbf{c}} = \underline{\mathbf{c}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

Solution:

$$\underline{\mathbf{c}}(\omega) = -(A + \omega^2 I)^{-1} \underline{\mathbf{F}}_0 .$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

Exercise 4) Continuing with the example from Monday and today, and taking the case $k = m$, consider the special case of the previous page's discussion. Here we are forcing the second mass sinusoidally:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = 1$, $\omega_2 = \sqrt{3}$ and that

$$\underline{x}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{x}_p(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \rightarrow 1$ or $\omega \rightarrow \sqrt{3}$.

Solution: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{x} = \underline{x}_p + \underline{x}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \frac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \frac{6 - 3\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix} + C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Interpretation as far as inferred practical resonance for slightly damped problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)$ d term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the c(ω) term for the undamped problem....see next page for visualizations.

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> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-2, 1, 1, -2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> c :=  $\omega \rightarrow (A + \omega^2 \cdot \text{Iden})^{-1} \cdot (-F0)$  : # the formula we worked out by hand
> c( $\omega$ );

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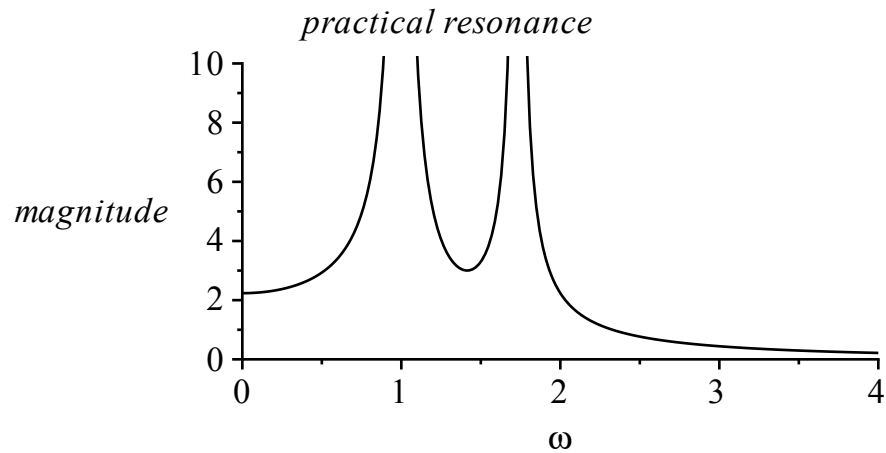
$$\begin{bmatrix} \frac{3}{3 - 4\omega^2 + \omega^4} \\ -\frac{3(-2 + \omega^2)}{3 - 4\omega^2 + \omega^4} \end{bmatrix}$$

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> with(plots) :
> with(LinearAlgebra) :
> plot(Norm(c( $\omega$ ), 2),  $\omega = 0..4$ , magnitude=0..10, color=black, title=`practical resonance`);
# Norm(c( $\omega$ ), 2) is the magnitude of the c( $\omega$ ) vector

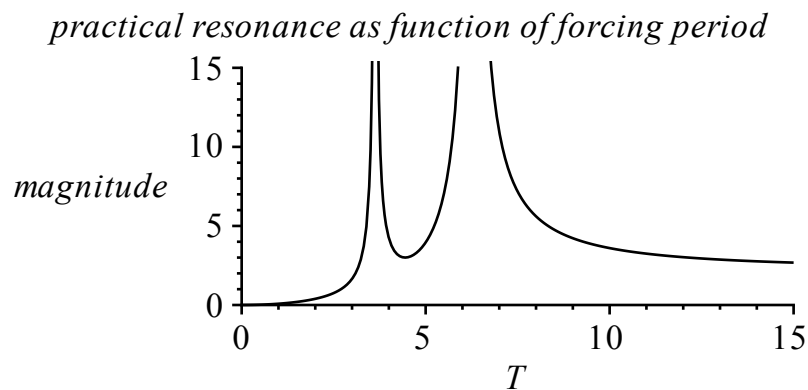
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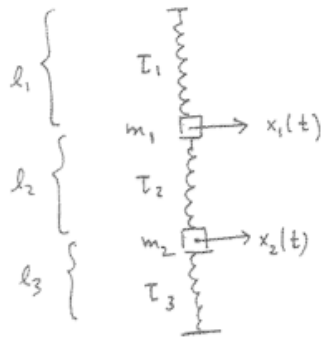
> plot(Norm(c( $\frac{2 \cdot \text{Pi}}{T}$ ), 2), T=0..15, magnitude=0..15, color=black, title
= `practical resonance as function of forcing period`);

```



There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions \perp to the mass-spring configuration)



T_1, T_2, T_3 are the tensions (forces) of the stretched springs ^{pulling}

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2 (x_2 - x_1) = -(K_1 + K_2) x_1 + K_2 x_2$$

$$m_2 x_2'' = K_2 (x_1 - x_2) - K_3 x_2 = K_2 x_1 - (K_2 + K_3) x_2$$

where K_1, K_2, K_3 are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2nd order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & ~~the~~ fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:



horiz force from top spring on mass 1

$$= -T_1 \sin \theta_1 = -T_1 \frac{x_1}{\sqrt{l_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{l_1} = -\frac{T_1}{l_1} x_1$$

$$\text{So } K_1 = \frac{T_1}{l_1}$$

$$\text{similarly, } K_2 = \frac{T_2}{l_2}, K_3 = \frac{T_3}{l_3}$$

for our physics demo springs, equilibrium length ≈ 0 , very Hookean so $T \approx k l$; $\frac{T}{l} \approx k$, so actually almost recover same $\frac{l}{l}$ fundamental modes !!

http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)