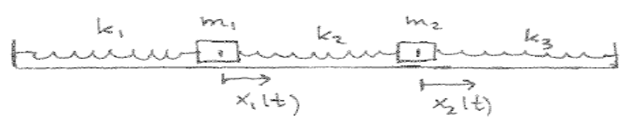
Math 2250-1
Mon Nov 26

First, Finish section 7.3 material, by considering the non-homogeneous first order system arising from the input-output model in Wednesday's notes - this was Exercise 5. Then proceed to section 7.4:

7.4 Mass-spring systems. Case 1: undamped and unforced oscillations.

In your homework for last week you modeled the spring system below, with no damping. Although we draw the picture horizontally, it would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field. In fact, we'll do an experiment tomorrow with just such a vertical configuration.



Let's make sure we understand why the natural system of DEs and IVP for this system is

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why?

1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why?

We can write the system of DEs in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$).

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}.$$

(You can think of A as the "acceleration" matrix.)
Notice that the simplification above is mathematically identical to the algebraic operation of multiplying by preceding matrix equation by the (diagonal) inverse of the diagonal mass matrix M:

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} \quad \Rightarrow \underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ , \text{with } A = M^{-1}K.$$

<u>How to find a basis for the solution space to conserved-energy mass-spring systems of DEs</u>

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

Notice that this is a system of second order DE's. We could convert it to a system of twice as many first order DE's, and then use the methods of section 7.3. But we'll take a shortcut, based on what worked for undamped oscillators and simple harmonic motion in Chapter 5. The simplest solutions to this homogeneous system would be of the form $f(t)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is a constant vector. In the case of a single mass, we got simple harmonic motion spanned by sinusoidal functions $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where ω_0 depended on k, m. We first tried $e^{r\,t}$ but Euler's formula led us to the trig functions. You can do a similar derivation here, starting with guesses of the form $e^{\mu\,t}\underline{\boldsymbol{v}}$, but let's just cut to the chase and try right away for solutions of the form

$$\cos(\omega\,t)\underline{\boldsymbol{v}} \quad \sin(\omega\,t)\underline{\boldsymbol{v}}\ .$$

If we substitute $\underline{\boldsymbol{x}}(t) = \cos(\omega\,t)\underline{\boldsymbol{v}}$ in the DE system we get

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} \quad \Rightarrow -\omega^2\cos(\omega\,t)\underline{\boldsymbol{v}} = A(\cos(\omega\,t)\underline{\boldsymbol{v}}) = \cos(\omega\,t)A\,\underline{v}\ .$$

This identity will hold $\forall\ t$ if and only if

$$A\,\underline{\boldsymbol{v}} = -\omega^2\underline{\boldsymbol{v}}.$$

So, $\underline{\boldsymbol{v}}$ must be an eigenvector of A, but its eigenvalue is $\lambda = -\omega^2$. If we used a trial solution $\underline{\boldsymbol{y}}(t) = \sin(\omega\,t)\underline{\boldsymbol{v}}$ we would arrive at the same eigenvector equation. This leads to the

<u>Solution space algorithm:</u> Consider a <u>very special case</u> of a homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

<u>If $A_{n\times n}$</u> is a diagonalizable matrix and <u>if all of its eigenvalues are negative</u>, then for each <u>eigenpair</u> $(\lambda_j, \underline{\boldsymbol{v}}_j)$ there are two linearly independent solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\underline{\boldsymbol{x}}_j(t) = \cos(\omega_j\,t)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin(\omega_j\,t)\underline{\boldsymbol{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the diagram on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity contribution to the solution space, $(c_1 + c_2\, t)\underline{\mathbf{v}}$, where $\underline{\mathbf{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the 4 − dimensional solution space to this two-mass, three-spring system.

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1\cos\left(\omega_1 t - \alpha_1\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\cos\left(\omega_2 t - \alpha_2\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1\cos\left(\omega_1 t\right) + c_2\sin\left(\omega_1 t\right)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3\cos\left(\omega_2 t\right) + c_4\sin\left(\omega_2 t\right)\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1,\quad x_1'(0) = a_2$$
$$x_2(0) = b_1,\quad x_2'(0) = b_2$$

Tomorrow: experiment with a two-mass, three spring configuration, to illustrate the two fundamental modes and to compare our linear model to actual data. Then, consider forced oscillation problems.