

Math 2250-1
 Tues Nov 20
 7.2-7.3

- First discuss the general theory for solving homogeneous and non-homogeneous linear systems of differential equations, in Monday's notes.
- Then carefully discuss how to use matrix eigenvalues and eigenvectors to find the general solution to first order homogeneous systems of differential equations, in case the matrix A does not depend on time:

$$\mathbf{x}'(t) = A \mathbf{x}$$

This discussion is in Monday's notes. The summary is that if $A_{n \times n}$ is diagonalizable, then we can find a basis of n solutions to the system above, of the form

$$\mathbf{x}_j(t) = e^{\lambda_j t} \mathbf{v}_j, \quad j = 1 \dots n$$

with $A \mathbf{v}_j = \lambda_j \mathbf{v}_j$.

The last example we worked on Monday illustrates this method. We used it to solve the initial value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

We found eigenvalues and eigenvectors for the matrix as follows:

$$\lambda = -1, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = -6, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

So the general solution to the system of DEs is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

To solve the initial value problem for this system of DEs we solved the linear system

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

to get $c_1 = 2, c_2 = -1$, i.e.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-6t} \\ -2e^{-t} + 6e^{-6t} \end{bmatrix}.$$

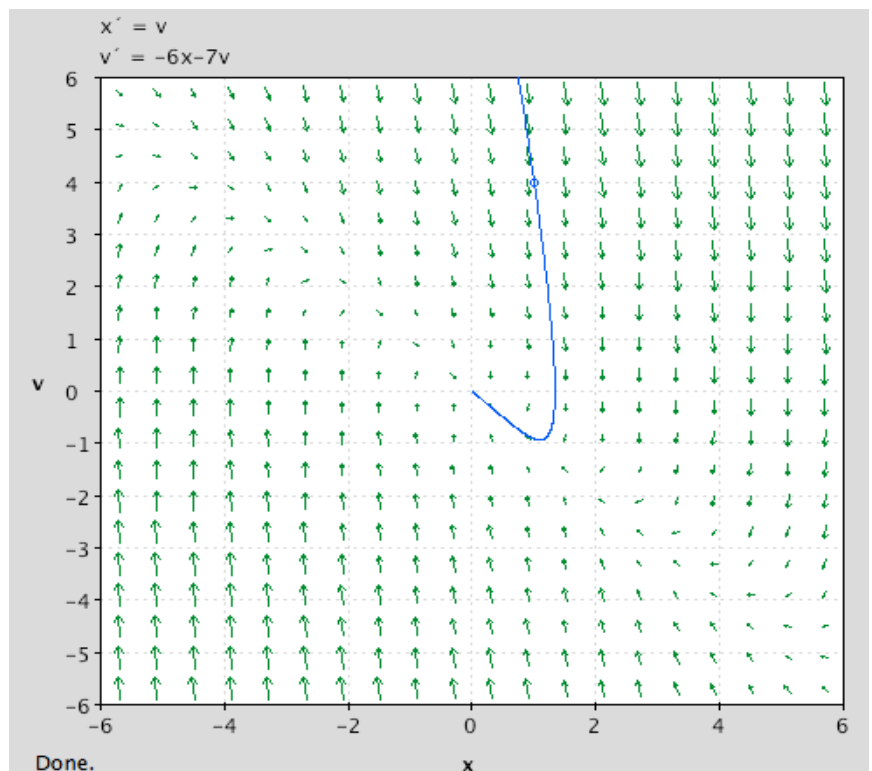
Exercise 1a) What second order overdamped initial value problem is equivalent to the first order system IVP on the previous page. And what is the solution function to this IVP?

b) What do you notice about the Chapter 5 "Wronskian matrix" for the second order DE in 1a, and the Chapter 7 "Wronskian matrix" for the solution to the equivalent first order system?

c) Since in the correspondence above, $x_2(t)$ equals the mass velocity $x'(t) = v(t)$, I've created the pplane phase portrait below using the lettering $[x(t), v(t)]^T$ rather than $[x_1(t), x_2(t)]^T$. Interpret the behavior of the overdamped mass-spring motion in terms of the pplane phase portrait.

d) How do the eigenvectors show up in the phase portrait, in terms of the direction the origin is approached from as $t \rightarrow \infty$, and the direction solutions came from (as $t \rightarrow -\infty$)?

<http://math.rice.edu/~dfield/dfpp.html>



So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is \mathbb{C}^n rather than \mathbb{R}^n .

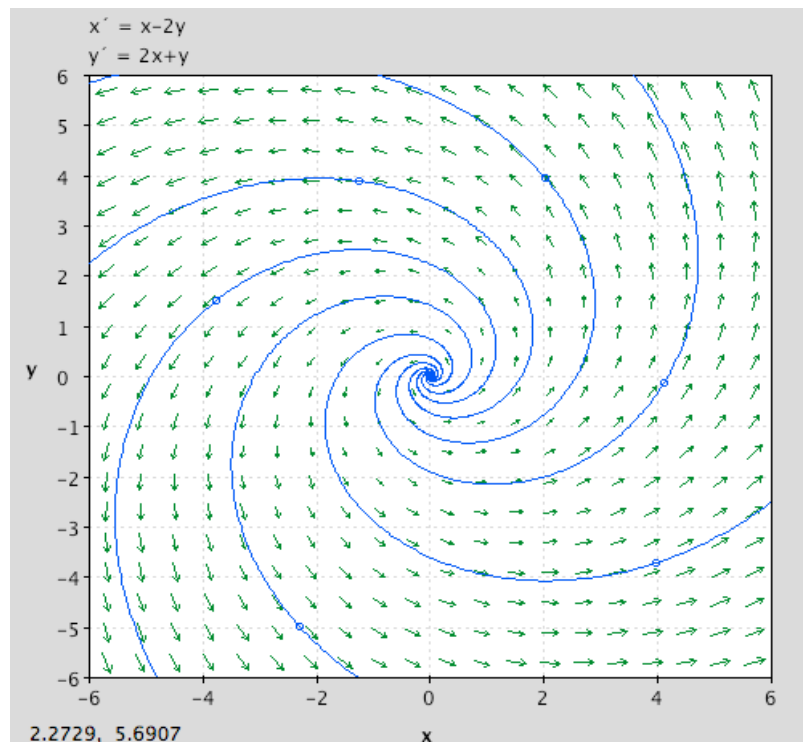
Definition: $\underline{v} \in \mathbb{C}^n$ ($\underline{v} \neq \underline{0}$) is a complex eigenvector of the matrix A , with eigenvalue $\lambda \in \mathbb{C}$ if $A \underline{v} = \lambda \underline{v}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. We can also refer to the general discussion on the following page, at appropriate stages.

Exercise 2) Find the general solution to the following homogeneous system of DEs. Interpret your solution in terms of the pplane phase portrait shown below

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$



Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b i \in \mathbb{C}$$

$$\underline{\mathbf{y}} = \underline{\alpha} + i \underline{\beta} \in \mathbb{C}^n$$

satisfy $A \underline{\mathbf{y}} = \lambda \underline{\mathbf{y}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\mathbf{z}}(t) = e^{\lambda t} \underline{\mathbf{y}}$ is a complex solution to

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

because $\underline{\mathbf{z}}'(t) = \lambda e^{\lambda t} \underline{\mathbf{y}}$ and this is equal to $A \underline{\mathbf{z}} = A e^{\lambda t} \underline{\mathbf{y}} = e^{\lambda t} A \underline{\mathbf{y}}$.

- But if we write $\underline{\mathbf{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i \underline{\mathbf{y}}(t)$$

then the equality

$$\begin{aligned} \underline{\mathbf{z}}'(t) &= A \underline{\mathbf{z}} \\ \Rightarrow \underline{\mathbf{x}}'(t) + i \underline{\mathbf{y}}'(t) &= A(\underline{\mathbf{x}}(t) + i \underline{\mathbf{y}}(t)) = A \underline{\mathbf{x}}(t) + i A \underline{\mathbf{y}}(t). \end{aligned}$$

Equating the real and imaginary parts on each side yields

$$\begin{aligned} \underline{\mathbf{x}}'(t) &= A \underline{\mathbf{x}}(t) \\ \underline{\mathbf{y}}'(t) &= A \underline{\mathbf{y}}(t) \end{aligned}$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A(\underline{\alpha} + i \underline{\beta}) = (a + b i)(\underline{\alpha} + i \underline{\beta})$ then it is straightforward to check that $A(\underline{\alpha} - i \underline{\beta}) = (a - b i)(\underline{\alpha} - i \underline{\beta})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i b)t}(\underline{\alpha} - i \underline{\beta}) = \underline{\mathbf{x}}(t) - i \underline{\mathbf{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).