Math 2250-1
Week 9 concepts and homework, due October 26.

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

5.3: using the algorithm for finding the general solution to constant coefficient linear homogeneous differential equations: real roots, Euler's formula and complex roots, repeated roots; solving associated initial value problems.

5.3: 3, 9, 11, 23, 27.

w9.1) Do the following problems by hand. Check your work with Maple (or other software). Hand in a printout of your computer verifications, along with your written work.
a) Find the general solution to the differential equation for $y(x)$
$$2\,y'' + 9\,y' - 5\,y = 0\,.$$
Hint: In this example you can factor the characteristic polynomial by using "f-o-i-l" backwards. Of course, you could also use the quadratic formula.
b) Find the general solution to the differential equation for $x(t)$
$$x'' + 8\,x' + 41\,x = 0\,.$$
Hint: completing the square works well here - probably better than the quadratic formula.
c) Solve the initial value problem for the differential equation in **b**, with $x(0) = 2, x'(0) = -2$.
d) Find the general solution to the differential equation for $y(x)$
$$y''' - y = 0.$$
e) Find the general solution to the differential equation for $y(x)$
$$y^{(6)} - 4\,y^{(4)} + 4\,y^{(2)} = 0.$$
w9.2) Euler's formula
$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)$$
is extremely useful in higher mathematics/science/engineering. In class we discuss how this definition is motivated by Taylor series. Amazingly, the rule of exponents is true for such expressions. In other words
$$e^{i\,\alpha + i\,\beta} = e^{i\,\alpha}\,e^{i\,\beta}\,.$$
Check this identity by rewriting the left side as $e^{i\,(\alpha + \beta)}$ and then using Euler's formula to expand both sides. You'll notice that the identity is true because of the addition angle formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. (And so this gives a good way to recover the trig. identities if you happen to forget them.)

w9.3) We explore why the general algorithm for finding a basis of solutions for constant coefficient homogeneous linear differential equations yields independent functions, in the case of an illustrative example.
a) What 4th order constant coefficient homogeneous linear DE has characteristic polynomial
$$p(r) = (r - 3)^2(r^2 + 4)\ ?$$
b) According to our general algorithm, write down a basis for the 4-dimensional solution space to the DE in part **a**.
c) Prove that the four functions you wrote down in part **b** actually are linearly independent (so automatically span, so a basis). Hint: Use a combination of the techniques we discussed on Wednesday October 17 and Friday October 19.

In the next exercise you will continue to explore the reasoning that leads to the "magic" general algorithm for solving constant coefficient homogeneous linear DEs in the case that the characteristic polynomial has repeated roots. These ideas are also discussed in section 5.3, and it may help you to read the book's discussion. Here's a summary: Begin with the derivative operator D defined by $D(y) := y'$. Then the second derivative operator is $D^2(y) := D \circ D(y)$, etc. So for $y = y(x)$ and

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

we may write

$$L = D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0 I$$

where I is the identity operator, $I(y) := y$. In other words, we are writing the operator L as a linear combination of various derivative operators. It turns out that any factorization of the characteristic polynomial $p(r)$ leads to a corresponding factorization of L. For example, consider the second order case with real roots,

$$L = D^2 + a_1 D + a_0 I .$$

If $p(r)$ factors with real roots,

$$p(r) = r^2 + a_1 r + a_0 = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1 r_2$$

Then L factors the same way, as a composition of first order operators:

$$L = D^2 - (r_1 + r_2)D + r_1 r_2 I = (D - r_1 I) \circ (D - r_2 I).$$

(To check this is to work from right to left above:

$$(D - r_1 I) \circ (D - r_2 I) = D \circ (D - r_2 I) - r_1 I \circ (D - r_2 I)$$
$$= D \circ D - D \circ r_2 I - r_1 I \circ D - r_1 I \circ (-r_2 I)$$
$$= D^2 - r_2 D - r_1 D - r_1 r_2 I = L .)$$

Now, compute

$$(D - r_1 I)\left(e^{r_1 x}\right) = D\left(e^{r_1 x}\right) - r_1 e^{r_1 x} = r_1 e^{r_1 x} - r_1 e^{r_1 x} = 0.$$

Thus this factoring of L as

$$L = (D - r_1 I) \circ (D - r_2 I) = (D - r_2 I) \circ (D - r_1 I)$$

is (another reason) why $e^{r_1 x}, e^{r_2 x}$ both satisfy $L(y) = 0$ in case $r_1 \neq r_2$ are roots of the characteristic polynomial, since each exponential is transformed into the zero function by one of the linear factors of L. The case $r_1 = r_2$ is the double root case,

$$L = (D - r_1 I) \circ (D - r_1 I) = (D - r_1 I)^2 .$$

More generally, if the characteristic polynomial for the nth order homogeneous DE factors as

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2} \ldots (r - r_m)^{k_m}$$

then the corresponding operator L factors as a composition

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m} .$$

w9.4)
a) Let $f(x)$ be any differentiable function. Check that

$$(D - r_1 I)\left(f(x)e^{r_1 x}\right) = f'(x)e^{r_1 x} .$$

b) Deduce that

$$\left(D - r_1 I\right) e^{r_1 x} = 0$$

$$\left(D - r_1 I\right) \left(x e^{r_1 x}\right) = e^{r_1 x}$$

$$\left(D - r_1 I\right) x^2 e^{r_1 x} = 2\, x e^{r_1 x}.$$

c) Explain why if L has a factor

$$\left(D - r_1 I\right)^{k_1}$$

with $k_1 > 1$, then the k_1 functions

$$e^{r_1 x},\ x e^{r_1 x},\ \dots,\ x^{r_1 - 1} e^{r_1 x}$$

as we wrote down in the algorithm for finding a solution space basis, do all indeed satisfy $L(y) = 0$.

5.4: mechanical vibrations: setting up and solving differential equations and initial value problems for mechanical vibrations; underdamped, critically damped, and overdamped vibration problems; applications to pendulums, mass-spring configurations, and related problems; understanding and using phase-amplitude form for sinusoidal functions.

5.4: 3, 4, 5, **6, 8,** (except in **8** assume the clock actually gains 12 minutes per week, rather than losing 10 minutes per day.) **10,** (in **10** the weight of the buoy is $m\, g = \rho\, V g = \rho\, \pi\, r^2\, h\, g$. The text forgot to include the density ρ in computing the mass from the volume), 15, 17, 21, **23**.

w9.5) Consider an unforced vibrating mass-spring system in which the mass is 3 kg, the spring has Hooke's constant 48 $\dfrac{N}{m}$ and we are able to vary the damping coefficient c. Thus the differential equation governing the displacement $x(t)$ m from equilibrium is
$$3\, x''(t) + c\, x'(t) + 48\, x(t) = 0.$$
As we vary c we will keep the initial values the same:
$$x(0) = 2$$
$$x'(0) = 8.$$

w9.5a) Solve the initial value problem when there is no damping, i.e. $c = 0$. Convert you solution to phase-amplitude form.

w9.5b) Now consider an underdamped configuration, when $c = 0.6$. Solve the same initial value problem as before, by hand. Use decimal approximations (with five digits) rather than using exact algebra. Check your solution with technology, e.g. Maple. Then convert your solution into pseudo-amplitude, pseudo-phase form.

w9.5c) What value of c leads to critical damping? Solve the IVP for this value of c

w9.5d) Finally, consider the overdamped situation, when $c = 30$. Solve the initial value problem in this case.

w9.5e) Use Maple or other software to create a display containing the graphs of all four solutions above, on the interval $0 \le t \le 4$. Print out a copy, and label which graph corresponds to which solution. Also write on it to indicate the amplitude, period, and time delay for the solution to the undamped problem **a**. Indicate the pseudo-period of the underdamped solution **b**, for example as the distance between two appropriate $t-$ intercepts. (If you're really ambitious you can do all of this labeling by creating a key in Maple and by adding various other items to your display, but this is definitely not required.)