Math 2250-1
Week 8 concepts and homework, due October 19.

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

4.1-4.4 review problem:
w8.0) Consider matrix $A_{3 \times 4}$ given by

$$A := \begin{bmatrix} 1 & 3 & -2 & 0 \\ -1 & -3 & 2 & 0 \\ 2 & 6 & -3 & 4 \end{bmatrix}.$$

The reduced row echelon of this matrix is

$$\begin{bmatrix} 1 & 3 & 0 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

w8.0a) Find a basis for the homogeneous solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^4 \ s.t. \ A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$. What the dimension of this subspace?
w8.0.b) Find a basis for the span of the columns of A, which is a subspace of $\mathbb{R}^3$. What is the dimension of this subspace?
w8.0c) The dimensions of the two subspaces in parts a,b add up to 4, the number of columns of A. This is an example of a general fact, known as the "rank plus nullity theorem". To see why this is always true, consider any matrix $B_{m \times n}$ which has m rows and n columns. As in parts a,b consider the homogeneous solution space

$$W = \{\underline{\mathbf{x}} \in \mathbb{R}^n \ s.t. \ B\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\} \subseteq \mathbb{R}^n$$

and the column space

$$V = span\{col_1(B), col_2(B), ...col_n(B)\} = \{B\,\underline{\mathbf{x}}, \ s.t. \ \underline{\mathbf{x}} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Let the reduced row echelon form of B have k leading 1's, with $0 \le k \le n$. Explain what the dimensions of W and V are in terms of k and n, and then verify that

$$dim(W) + dim(V) = n$$

must hold. (The dimension of the column space V is called the rank of the matrix, and the dimension of the homogeneous solution space W is called the nullity of the matrix, which is why this is called the "rank plus nullity theorem".)

5.1
Solving initial value problems for linear homogeneous second order differential equations, given a basis for the solution space. Finding general solutions for constant coefficient homogeneous DE's by searching for exponential or other functions. Superposition for linear differential equations, and its failure for non-linear DE's.

1, **6**, (in 6 use initial values $y(0) = 4$, $y'(0) = -2$ rather than the ones in the text), **10**, 11, **12**, **14** (In 14 use the initial values $y(1) = 2, y'(1) = -4$ rather than the ones in the text.), 17, **18, 27**, 33, 39.

w8.1) In 5.1.6 above, the text tells you that $y_1(x) = e^{2x}, y_2(x) = e^{-3x}$ are two independent solutions to the second order homogeneous differential equation $y'' + y' - 6y = 0$. Verify that you could have found these two exponential solutions via the following guessing algorithm: Try $y(x) = e^{rx}$ where r is to be determined. Substitute this possible solution into the homogeneous differential equation and find the only two values of r for which $y(x)$ will satisfy the DE.

5.2 Testing collections of functions for dependence and independence. Solving IVP's for homogeneous and non-homogeneous differential equations. Superposition.
1, 2, 5, **8**, 11,13, **16**, 21, **25**, 26

Here's a problem that integrates many of the ideas from sections 5.1-5.2, and also ties in with chapter 4:
w8.2) Consider the three functions

$$y_1(x) = \sin(2x), \quad y_2(x) = \sin\left(2\left(x - \frac{\pi}{4}\right)\right), y_3(x) = \cos(2x).$$

a) Show that all three functions solve the differential equation

$$y'' + 4y = 0.$$

b) The differential equation above is a second order linear homogeneous DE, so the solution space is 2-dimensional. Thus the three functions y_1, y_2, y_3 above must be linearly dependent. Find a linear dependency. (Hint: use a trigonometry addition angle formula.)

c) Explicitly verify that every initial value problem

$$y'' + 4y = 0$$
$$y(0) = b_1$$
$$y'(0) = b_2$$

has a solution of the form $y(x) = c_1\sin(2x) + c_2\cos(2x)$, and that c_1, c_2 are uniquely determined by b_1, b_2. Use the existence-uniqueness theorem to explain why this proves that $y_1(x)$ and $y_3(x)$ are a basis for the solution space to

$$y'' + 4y = 0.$$

d) Compute the Wronskian matrix for $y_1(x), y_3(x)$. What does this matrix, evaluated at $x = 0$, have to do with the algebra related to **c**? What is the significance of the fact that the Wronskian (determinant) is non-zero at $x = 0$, in terms of knowing that c_1, c_2 are uniquely determined by b_1, b_2?

e) Find by inspection, particular solutions $y(x)$ to the two non-homogeneous differential equations

$$y'' + 4y = 16, \quad y'' + 4y = -8x$$

Hint: one of them could be a constant, the other could be a multiple of x.

f) Use superposition and your work from **c,e** to find the general solution to the non-homogeneous differential equation

$$y'' + 4y = 16 - 8x.$$

g) Solve the initial value problem

$$y'' + 4y = 16 - 8x$$
$$y(0) = 0$$
$$y'(0) = 0.$$