Name Solutions

Student I.D. ______

Math 2250-1
Exam 2
November 8 2012

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. There is a Laplace transform table at the end of this test. **Good Luck!**

Score

		POSSIBLE
1		25
2		25
3		25
4		15
5		10
TOTAL		100

1) Here is a matrix and its reduced row echelon form:

$$A := \left[\begin{array}{ccccc|c} 1 & -2 & -2 & 0 & 1 & 0 \\ 3 & -6 & -5 & 1 & 4 & 0 \\ -2 & 4 & 2 & -2 & -4 & 0 \end{array}\right] \quad \text{reduced row echelon form of A:} \quad \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1a) Find the general solution to the homogeneous matrix equation $A\underline{x} = \underline{0}$. Write your solution in linear combination form.

(10 points)

back solve

$$\begin{aligned} x_1 &= 2p - 2q - 3t \\ x_2 &= p \\ x_3 &= -q - t \\ x_4 &= q \\ x_5 &= t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = p \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

1b) What is the dimension of the solution space to $A\underline{x} = \underline{0}$ that you found in 1a? Explain.

(5 points)

dim = $\underline{3}$, because there are $\underline{3}$ vectors in a basis, e.g. from above we may choose

basis $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

(These vectors span the sol'n space, and they are linearly indep.)

1c) The span of the five columns of the matrix A is a certain subspace of $\mathbb{R}^3$. Find a basis for this subspace, and describe what the subspace is geometrically.

(10 points)

For your convenience, here is the matrix and its reduced form, from page 1:

$$A := \begin{array}{c} \begin{bmatrix} 1 & -2 & -2 & 0 & 1 \\ 3 & -6 & -5 & 1 & 4 \\ -2 & 4 & 2 & -2 & -4 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{v}_1, & \vec{v}_2, & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix} \end{array}; \quad \text{reduced row echelon form of A: } \begin{array}{c} \begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \\ \vec{w}_1, & \vec{w}_2, & \vec{w}_3, & \vec{w}_4, & \vec{w}_5 \end{matrix} \end{array}.$$

Since the column dependencies of A and rref(A) are the same, we deduce

$$\vec{v}_2 = -2\vec{v}_1$$
$$\vec{v}_4 = 2\vec{v}_1 + \vec{v}_3$$
$$\vec{v}_5 = 3\vec{v}_1 + \vec{v}_3.$$

Thus, span of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$ is actually just the span of $\vec{v}_1, \vec{v}_3$.

Since $\vec{v}_1, \vec{v}_3$ are linearly independent, they are a basis for the span of the original 5 vectors.

Thus, this subspace is a (2-dimensional) <u>plane through the origin in $\mathbb{R}^3$.</u>

2a) Consider the linear homogeneous differential equation for $y(x)$:

$$y'''(x) - 4y''(x) + 4y'(x) = 0.$$

Find a basis for the solution space to this DE.

(10 points)

$$p(r) = r^3 - 4r^2 + 4r = r(r^2 - 4r + 4) = r(r-2)^2 \quad \text{has roots } r = 0, 2$$

$$\text{thus } y_H(x) = c_1 e^{0x} + c_2 e^{2x} + c_3 x e^{2x}$$
$$= c_1 + c_2 e^{2x} + c_3 x e^{2x}$$

and the 3 functions $1, e^{2x}, xe^{2x}$ are a basis.

2b) Use superposition and the method of undetermined coefficients to find the general solution $y(x)$ to

$$L(y) := y'''(x) - 4y''(x) + 4y'(x) = 16x + e^x.$$

(15 points)

for $L(y_1) = 16x$

try $y_1(x) = x(Ax + B)$

because $p(r)$ has a factor $(r-0)^1$.

$$\begin{aligned}
+0(\,y_1 &= Ax^2 + Bx\,)\\
+4(\,y_1' &= 2Ax + B\,)\\
-4(\,y_1'' &= 2A\,)\\
1(\,y_1''' &= 0\,)
\end{aligned}$$

$$L(y) = x(8A) + 1(4B - 8A) \overset{\text{want}}{=} 16x + 0$$

$$\Rightarrow 8A = 16 \Rightarrow A = 2$$
$$4B - 8A = 0 \Rightarrow B - 2A = 0$$
$$A = 2 \Rightarrow B = 4$$

$$\Rightarrow y_1(x) = 2x^2 + 4x$$

for $L(y_2) = e^x$

try $y_2 = Ce^x$

$$\Rightarrow y_2''' - 4y_2'' + 4y_2' = Ce^x(1 - 4 + 4) = Ce^x, \text{ so}$$

take $C = 1$

i.e. $y_2(x) = e^x$

general soln for $L(y) = 16x + e^x$

is $y = y_P + y_H$
$= y_1 + y_2 + y_H$

$$\boxed{y(x) = 2x^2 + 4x + e^x + c_1 + c_2 e^{2x} + c_3 x e^{2x}}$$

3a) Use Chapter 5 techniques to solve the initial value problem for this forced damped harmonic oscillator differential equation:

$$x''(t) + 4\,x'(t) + 13\,x(t) = 40\sin(3\,t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

You may use the fact that this DE has a particular solution
$$x_p(t) = -3\cos(3\,t) + \sin(3\,t)\,.$$
(You do not need to check this fact; you can just use it.)

(15 points)

for $x'' + 4x + 13x = 0$

$p(r) = r^2 + 4r + 13 = (r+2)^2 + 9$

has roots $(r+2)^2 = -9 \Rightarrow r+2 = \pm 3i \Rightarrow r = -2 \pm 3i$

$\Rightarrow x_H(t) = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$

$x(t) =$

$\Rightarrow x_p(t) + x_H(t) = -3\cos 3t + \sin 3t + c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$

$x(0) = 0 = -3 + c_1 \Rightarrow c_1 = 3$

$x'(t) = -9\sin 3t + 3\cos 3t + c_1(-2e^{-2t}\cos 3t + 3e^{-2t}\sin 3t) + c_2(-2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t)$

$\longrightarrow x'(0) = 0 = 3 - 2c_1 + 3c_2 \Rightarrow -3 + 3c_2 = 0 \Rightarrow c_2 = 1$

$$\boxed{\Rightarrow x(t) = -3\cos 3t + \sin 3t + 3e^{-2t}\cos 3t + e^{-2t}\sin 3t}$$

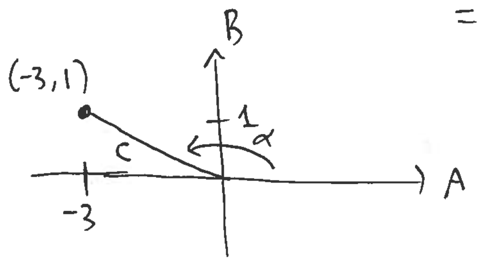
3b) Identify the "steady periodic" and "transient" parts of your solution to 3a. Then express the steady periodic part in amplitude-phase form. (Express the phase angle α as an inverse trig function; you don't need a decimal value.)

(10 points)

$x_{sp}(t) = -3\cos 3t + \sin 3t$

$x_{tr}(t) = 3e^{-2t}\cos 3t + e^{-2t}\sin 3t$

$$x_{sp}(t) = \underbrace{-3}_{A}\cos 3t + \underbrace{\sin 3t}_{B=1}$$
$$= C\cos(3t - \alpha)$$



$$\boxed{\begin{array}{l} C = \sqrt{A^2+B^2} = \sqrt{10} \\ \alpha = \arccos\left(\frac{-3}{\sqrt{10}}\right) \end{array}}$$

$\cos\alpha = \frac{A}{C} = \frac{-3}{\sqrt{10}}$

$\sin\alpha = \frac{B}{C} = \frac{1}{\sqrt{10}}$

4) Re-solve the IVP from problem 3,

$$x''(t) + 4x'(t) + 13x(t) = 40 \cdot \sin(3t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

using Laplace transform techniques (and the table at the end of the test). If you wish, you may use the particular solution $x_p(t) = -3\cos(3t) + \sin(3t)$ given in problem 3 to deduce two of the partial fraction coefficients for $X(s)$. This will save you time.

(15 points)

$$s^2X(s) + 4sX(s) + 13X(s) = \frac{120}{s^2+9}$$
$$X(s)(s^2+4s+13)$$

$$X(s) = \frac{120}{(s^2+4s+13)(s^2+9)} = \frac{As+B}{s^2+4s+13} + \frac{Cs+D}{s^2+9}$$

from #3, $x_p(t) = -3\cos 3t + \sin 3t$

$$\Rightarrow X_p(s) = \frac{-3s}{s^2+9} + \frac{3}{s^2+9} \Rightarrow C=-3, D=3 \text{ above}$$

partial fractions:

$$120 = (As+B)(s^2+9) + (-3s+3)(s^2+4s+13)$$

$$\begin{array}{l|l} & 1\,(9B+39) \Rightarrow 9B+39 = 120 \Rightarrow 9B = 81 \Rightarrow B=9 \\ +0s & +s(\\ +0s^2 & +s^2(\\ +0s^3 & +s^3(A-3) \Rightarrow A=3 \end{array}$$

$$\begin{array}{r} 120 \\ -39 \\ \hline 81 \end{array}$$

So $$X(s) = \frac{3s+9}{s^2+4s+13} - \frac{3s}{s^2+9} + \frac{3}{s^2+9}$$

$$X(s) = \frac{3(s+2)+3}{(s+2)^2+9} - 3\frac{s}{s^2+9} + \frac{3}{s^2+9}$$

$$\Rightarrow \boxed{x(t) = 3e^{-2t}\cos 3t + e^{-2t}\sin 3t - 3\cos 3t + \sin 3t}$$

5a) Use Laplace transforms (and the table you've been provided) to solve the forced oscillator initial value problem

$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

(8 points)

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2}$$

$$X(s)(s^2+\omega_0^2) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2} + s x_0 + v_0$$

$$X(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2} + x_0\frac{s}{s^2+\omega_0^2} + \frac{v_0}{\omega_0}\frac{\omega_0}{s^2+\omega_0^2}$$

$$\Rightarrow \boxed{x(t) = \frac{F_0}{m}\frac{t}{2\omega_0}\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

5b) What word describes the sort of behavior exhibited by solutions to the differential equation in 5a?

(2 points)

(pure) resonance!

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		