Name SOLUTIONS

Student I.D. ______________________

Math 2250-1
Exam #1
September 27, 2012

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not a "graphing calculator" i.e. not one which is capable of integration, taking derivatives, or matrix algebra. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

Score

		POSSIBLE
1		25
2		25
3		15
4		15
5		20
TOTAL		100

1) Consider a boat which starts at rest at time $t = 0$ sec, is accelerated in a straight path by an engine that provides a constant 1000 N of force, and which is also subject to drag forces of 40 N for each $\frac{m}{s}$ of velocity. The boat has mass 400 kg.

1a) Use your modeling ability to show that the boat velocity $v(t)$ (in meters per second) satisfies the initial value problem

$$v'(t) = 2.5 - .1\,v$$
$$v(0) = 0$$

(5 points)

$mv' = \text{Force}$

$400v' = 1000 - 40v$

$\div 400 \qquad v' = 2.5 - .1v$

1b) Solve the initial value problem in 1a.

(15 points)

$v' + .1v = 2.5$

$e^{.1t}(v' + .1v) = 2.5e^{.1t}$

$(e^{.1t}v)' = 2.5e^{.1t}$

$e^{.1t}v = \int 2.5e^{.1t}dt$

$e^{.1t}v = 25e^{.1t} + C$

$\div e^{.1t}: \quad v = 25 + Ce^{-.1t}$

@ $t = 0$: $0 = 25 + C$

$\Rightarrow C = -25.$

$$\boxed{v(t) = 25 - 25e^{-.1t}}$$

1c) How far does the boat travel in the first 20 seconds?

(5 points)

$x(20) - x(0) = \int_0^{20} v(t)dt$

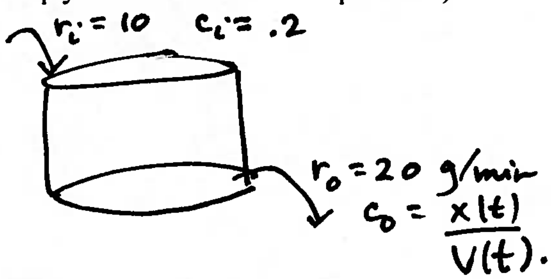
$= \int_0^{20} 25 - 25e^{-.1t}dt = 25t + 250e^{-.1t}\Big]_0^{20}$

$= 25\cdot 20 + 250(e^{-2} - 1)$

$= 500 - 250 + 250e^{-2}$ m.

$(\approx 284\text{ m})$

2) Consider the following input-output model: A brine tank initially contains 1000 gallons of very salty water - the initial concentration is 1 *lb* of salt for each gallon of water. At time $t = 0$ less salty water begins to flow into the tank, at a rate of 10 gallons per minute. This water contains 0.2 pounds of salt per gallon. Also at time $t = 0$ water begins to flow out of the tank, at 20 gallons per minute. (There is space below for you to sketch the configuration, as it may help you with the rest of the problem.)



2a) When does the tank become empty?

(5 points)

$r_i - r_o = 10 - 20 = -10$ g/min.

since $V_0 = 1000$ it will take 100 minutes to drain 1000 gallons.

2b) Find a formula for the volume $V(t)$ of water in the tank at time t minutes, until the time of emptying.

(5 points)

$V'(t) = -10 \quad (r_i - r_o)$

$V(0) = 1000$

$\Rightarrow \boxed{V(t) = 1000 - 10t}$

2c) Use your modeling ability to justify why the initial value problem for the pounds $x(t)$ of salt in the tank at time t minutes turns out to be

$$x'(t) + \frac{2}{100 - t} x(t) = 2$$
$$x(0) = 1000$$

(until the tank empties).

(5 points)

$$x' = r_i c_i - r_o c_o$$
$$= 10(.2) - 20 \frac{x(t)}{V(t)}$$
$$= 2 - 20 \frac{x}{1000 - 10t}$$
$$x' = 2 - \frac{2x}{100 - t}$$

or $\boxed{x' + \frac{2}{100-t} x = 2}$

since initial concentration is 1 lb/gal. & $V_0 = 1000$ g,

$\boxed{x(0) = 1 \cdot 1000 = 1000 \text{ lb}}$

2d) Solve the IVP in 2c (until the tank empties). For your convenience, the IVP is repeated here:

$$x'(t) + \frac{2}{100-t}x(t) = 2$$
$$x(0) = 1000$$

(10 points)

$$\int \frac{2}{100-t}\,dt = -2\ln(100-t)$$

$$e^{-2\ln(100-t)} = (100-t)^{-2}$$

$$(100-t)^{-2}\left(x' + \frac{2}{100-t}x\right) = 2(100-t)^{-2}$$

$$\frac{d}{dt}\left[(100-t)^{-2}x\right] = 2(100-t)^{-2}$$

$$(100-t)^{-2}x = \int 2(100-t)^{-2}\,dt = 2(100-t)^{-1} + C$$

$\cdot (100-t)^2$:

$$x = 2(100-t) + C(100-t)^2$$

$$x(0) = 1000 \Rightarrow 1000 = 200 + C \cdot 10{,}000$$

$$\frac{800}{10{,}000} = C = .08$$

$$x(t) = 2(100-t) + .08(100-t)^2$$

3) Consider the differential equation for $x(t)$:

$$x'(t) = x^4 - 4x^2.$$

3a) Find the equilibrium solutions.

$$= x^2(x^2-4) = x^2(x-2)(x+2)$$

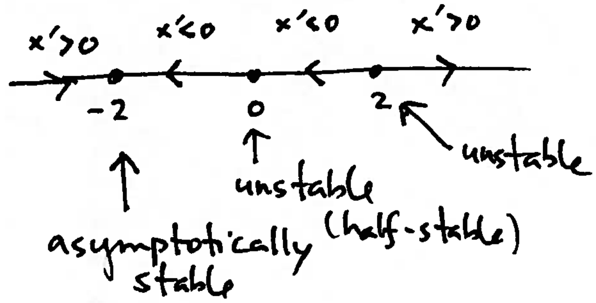
(3 points)

$$\boxed{x \equiv 0, 2, -2}$$

3b) Construct the phase diagram for this differential equation, and use it to determine the stability of the equilibrium solutions.

$$x' = x^2(x-2)(x+2)$$

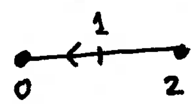
(7 points)



3c) Let $x(t)$ be the solution to the initial value problem for this differential equation, with $x(0) = 1$. Without solving for $x(t)$ deduce the value of $\lim_{t\to\infty} x(t)$. Explain.

(5 points)

part of phase diagram.



since $0 < 1 < 2$ and since $x'(t) < 0$ on $(0,2)$,

$$\lim_{t\to\infty} x(t) = 0.$$

4a) Use Cramer's rule to solve the linear system

$$x + 3y = -1$$
$$2x + 5y = -1 .$$

(6 points)

$$x = \frac{\begin{vmatrix} -1 & 3 \\ -1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{-2}{-1} = 2$$

$$y = \frac{\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{1}{-1} = -1$$

check: $2 - 3 = -1$ ✓
$4 - 5 = -1$ ✓

4b) Solve the matrix equation below for X:

$$X\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(9 points)

$$XA = B$$

mult. by A^{-1} on right:

$$XAA^{-1} = BA^{-1}$$
$$XI = BA^{-1}$$
$$X = BA^{-1}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A}\operatorname{Adj}(A)$$
$$= \frac{1}{-1}\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

check. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓.

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\boxed{X = \begin{bmatrix} -1 & 1 \\ -7 & 5 \end{bmatrix}}$$

check: $\begin{bmatrix} -1 & 1 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

✓

5) Consider the matrix equation

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

5a) Compute the determinant of the matrix in this system.

(4 points)

e.g. top row: $\det = 1(9-1) - 2(6) - 2(-2)$

$= 8 - 12 + 4 = \boxed{0}$

5b) What can you deduce about the number of solutions to this homogeneous matrix equation, based on your computation in part 5a? Explain.

(4 points)

system automatically consistent, since RHS $= \vec{0}$.

$|A| = 0 \iff \text{rref}(A) \neq I \iff$ free parameter in sol.

$\iff \boxed{\infty\text{'ly many sols}}$

5c) Compute the reduced row echelon form of the augmented matrix for this problem, and backsolve to find the explicit solution set for the system.

(10 points)

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$-2R_1+R_2$
$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$-R_2$
$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$+R_2+R_3$
$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-2R_2+R_1$
$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rref

backsolve: $\boxed{\begin{aligned} x &= -4t \\ y &= 3t \\ z &= t \end{aligned}}$ $t \in \mathbb{R}$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ $t \in \mathbb{R}$

(line thru origin).

5d) What is a geometric interpretation of the solution set for this problem, in terms of intersecting planes?

(2 points)

3 planes intersecting in a (common) line.