

Math 2250-1

Wed Dec 5

9.4 Nonlinear systems of first order differential equations arising from second order mechanical oscillation problems.

- Discuss the freely rotating rigid-rod pendulum, Tuesday's notes. We set up the equivalent first order system in Exercise 2, and found the equilibrium solutions. Complete Exercise 3 there, then also this additional portion here:

3 continued) In terms of deciding the stability for the equilibrium solutions with $x = n\pi$, n even, which linearize to stable centers, and so are borderline for the non-linear system, there is a separation of variables trick that resolves the issue in this case. It relies on the Calculus fact (which we've used before) that

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

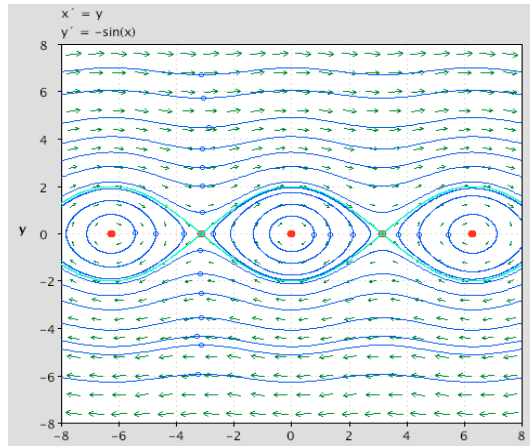
Thus, for the pendulum system,

$$\frac{dy}{dx} = - \frac{\frac{g}{L} \sin(x)}{y}.$$

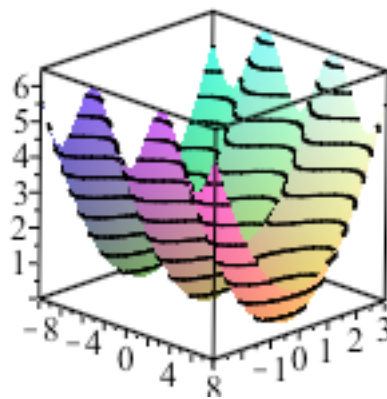
3c) Separate variables above, and use your work to show that the equilibrium solutions to the non-linear system are stable, when n is even.

3d) How is your work in c related to conservation of energy in the original pendulum problem?

3e) One way to visualize the results above is to create the 3-d graph of (a multiple) of the total energy function. In the case $\frac{g}{L} = 1$ and if we add a constant so that the total energy is zero at the stable equilibria, this is the function $f(x, y) = \frac{1}{2}y^2 + (1 - \cos(x))$. Notice that the contours correspond to the original pplane solution trajectories.

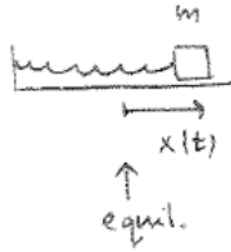


> with(plots) :
 $\text{plot3d}\left(1 - \cos(x) + \frac{1}{2} \cdot y^2, x = -8 \dots 8, y = -3 \dots 3, \text{axes} = \text{boxed}\right);$



>

Nonlinear springs: Consider our usual mass-spring configuration



only now, consider non-linear (but still autonomous) forcing functions:

$$m x''(t) = F(x) \\ = -kx + \alpha x^2 + \beta x^3 + \dots$$

The linear model, $F(x) = -kx$ assumes perfect "Hookean" springs. By considering more terms in a Taylor expansion as above, we can model different situations.

- If the forcing function is an odd function, it could be natural to use a model

$$m x'' = -kx + \beta x^3.$$

If $\beta < 0$ the return force is stronger than in the Hooke's case, i.e. a hard spring. If $\beta > 0$ the return force is weaker than expected, i.e. a soft spring.

- The forcing function might not be an odd function, in which case a model such as

$$m x'' = -kx + \alpha x^2$$

might be appropriate.

- We can convert any such spring equation

$$m x''(t) = F(x)$$

into the equivalent first order system for position and velocity,

$$x'(t) = v$$

$$v'(t) = \frac{F(x)}{m}.$$

Notice that the equilibrium (constant) solutions for the first order system, $v = 0$, $F(x) = 0$ correspond to the equilibrium (constant) solutions of the autonomous second order DE.

Exercise 1) Consider the soft spring problem

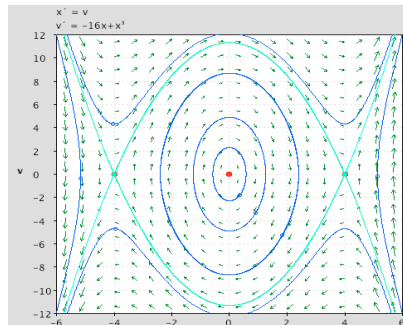
$$x''(t) = -16x + x^3.$$

1a) Convert this to an equivalent first order system of DEs.

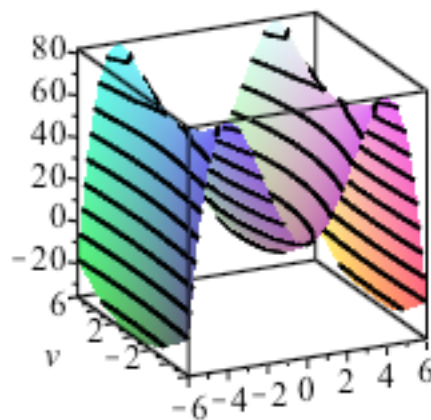
1b) Find the equilibrium solutions to the first order system of DEs, and note how they are equivalent to the equilibrium solutions to the second order DE we started from.

1c) Linearize the first order system at the equilibrium solutions, and discuss the stability (or borderline issues) at each equilibrium. Compare to the phase portrait below.

1d) Use a "total energy" computation - either physics or separation of variables will work, to resolve the borderline case.



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> with(plots):  
plot3d(1/2*v^2 + 8*x^2 - 1/4*x^4, x=-6..6, v=-6..6, axes=boxed);
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>
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why the complex eigenvalue with real part equal to zero in the linearized problem (stable center) is borderline for the non-linear problem:

For all three examples below, the linearization at the origin is

$$\begin{aligned} x'(t) &= y \\ y'(t) &= -x \end{aligned} \quad \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and the Jacobian matrix has eigenvalues $\lambda = \pm i$.

