

Math 2250-1

Tues Dec 4

9.2-9.4 Nonlinear systems of first order differential equations, with an emphasis on autonomous systems of two first order DEs. Linearization at equilibrium points, classification of equilibrium points, stability. Applications to interacting populations and mechanical systems.

We're discussing autonomous systems of two first order differential equations for $x(t), y(t)$:

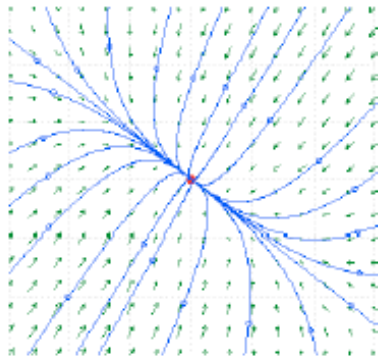
$$x' = F(x, y)$$

$$y' = G(x, y)$$

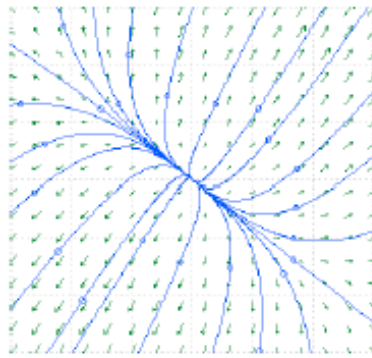
We're interested in equilibrium solutions, stability, linearization, the connections to phase portraits, and applications from interacting populations and mechanical oscillations.

- Yesterday we used a populations competition model from 9.3, to illustrate how to use the Jacobian matrix at equilibrium points to deduce the linearized system of DEs for (almost) solutions near equilibrium points.

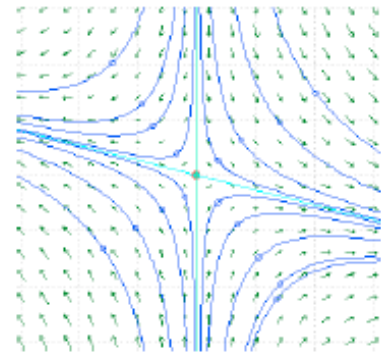
In the case the eigenvalues of the Jacobian matrix are real, we discussed the geometry of solution trajectories near stable and unstable nodes, and saddle points.... which also lets you discuss stability questions, at least for the linearized system of DEs.



nodal sink
 $\lambda_1, \lambda_2 < 0$



nodal source
 $\lambda_1, \lambda_2 > 0$



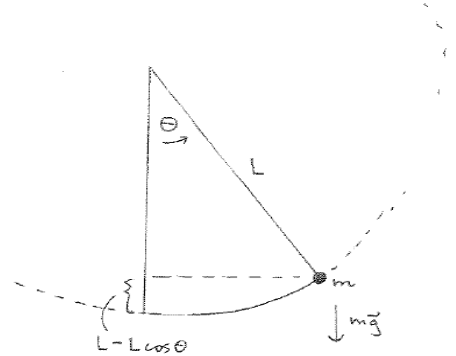
saddle point
 $\lambda_1 < 0 < \lambda_2$

Today we'll focus on linearization when the eigenvalues are complex.

- Return to Monday's notes, work through Exercise 2, and discuss the general classification theorem.

9.4) Interpreting non-linear mechanical oscillations in the phase plane. (We've already done this for linear undamped, underdamped and overdamped unforced oscillations, back in Chapter 7.)

A beautiful example of a non-linear mechanical system is the freely rotating rigid rod pendulum. We've already considered a special case of this configuration, when the angle θ from vertical is near zero. Now assume that the pendulum is free to rotate through any angle $\theta \in \mathbb{R}$.



Recall how we used conservation of energy before, to re-derive the dynamics for this (now) possibly rotating pendulum:

$$TE = KE + PE = \frac{1}{2} m v^2 + m g (L - L \cos(\theta))$$

$$TE = \frac{1}{2} m (L \theta'(t))^2 + m g L (1 - \cos(\theta(t))) .$$

(We use that the signed arclength $s(t)$ along the circular arc from vertical $\theta = 0$ to the mass position at general $\theta \in \mathbb{R}$ is given by $s(t) = L \theta(t)$, so that the rotational velocity is $v(t) = s'(t) = L \theta'(t)$.)

Since total energy is conserved in this undamped system,

$$\begin{aligned} \frac{d}{dt} TE &\equiv 0 \equiv m L^2 \theta'(t) \theta''(t) + m g L \sin(\theta(t)) \theta'(t) \\ &= m L \theta'(t) [L \theta''(t) + g \sin(\theta(t))] . \end{aligned}$$

Thus the dynamics of the free-to-rotate pendulum are given by the non-linear DE (that we saw before, and linearized near $\theta = 0$)

$$\begin{aligned} L \theta''(t) + g \sin(\theta(t)) &= 0 \\ \Rightarrow \theta''(t) + \frac{g}{L} \sin(\theta(t)) &= 0 . \end{aligned}$$

Exercise 1) Explain how the second order DE above is related to this first order system:

$$\begin{aligned} x'(t) &= y \\ y'(t) &= -\frac{g}{L} \sin(x) . \end{aligned}$$

Exercise 2a). Find the equilibrium solutions to the first order system

$$\begin{aligned}x'(t) &= y \\ y'(t) &= -\frac{g}{L} \sin(x)\end{aligned}$$

2b) How are the equilibrium solutions in a related to the equilibrium (i.e. constant) solutions to the second order autonomous differential equation

$$\theta''(t) + \frac{g}{L} \sin(\theta(t)) = 0 ?$$

Exercise 3) In the previous exercise you found that the equilibrium solutions are points along the x -axis of the form $(n\pi, 0)$ where $n \in \mathbb{Z}$.

3a) Show that if n is odd, then the equilibrium point is an (unstable) saddle point.

3b) Show that if n is even, then the equilibrium point is a stable center for the linear system, so indeterminate for the non-linear one. What's your guess for stability in the non-linear problem? Why? We'll continue this discussion tomorrow, but on the next page are some beautiful phase portraits that you should be able to interpret in terms of the physics. The first one represents the undamped rigid-rod pendulum, with $\frac{g}{L} = 1$. In the second picture the pendulum is slightly damped.

