

Math 2250-1

Fri Aug 24

1.3-1.4: slope fields; existence and uniqueness for solutions to IVPs; examples we can check with separation of variables.

- The discussion of slope fields and section 1.3 begins on page 2 of Wednesday's notes, and that is exactly where we left off at the end of class on Wednesday. So, we'll begin there today, after you take the quiz.

After we finish exercises 2,3 on Wednesday's notes, let's do

Exercise 1: In your homework for today you used the fact from 1.43a, that solutions $v = v(t)$ to the differential equation

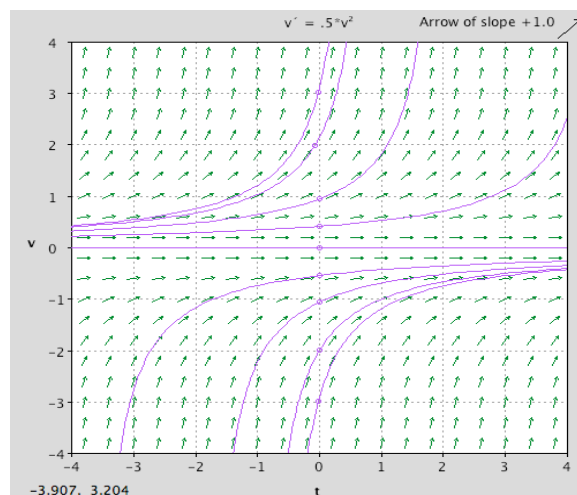
$$\frac{dv}{dt} = k v^2$$

(k a constant) are given by

$$v(t) = \frac{1}{C - k t} .$$

Derive these solutions using separation of variables. Notice that $v \equiv 0$ is also a solution. Can you guess why the separation of variables algorithm doesn't find it?

Here's a picture to go with our work above: I made it using the internet tool called "dfield" (for direction field). You should be able to find it with an easy google search....or you can use this link <http://math.rice.edu/~dfield/dfpp.html>



1.4 Separable DE's:

Since, as we've seen, a lot of the examples we study in 1.3 are separable DE's, let's take some time to discuss precisely what they are, and why the separation of variables algorithm works.

Definition: A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) .$$

It's more convenient to rewrite this DE as

$$\frac{1}{\phi(y)} \frac{dy}{dx} = f(x), \quad (\text{as long as } \phi(y) \neq 0) .$$

Writing $g(y) = \frac{1}{\phi(y)}$ the differential equation reads

$$g(y) \frac{dy}{dx} = f(x) .$$

Solution (math justified): Suppose $G(y)$ is an antiderivative with respect to y of $g(y)$ and $F(x)$ is an antiderivative of $f(x)$ with respect to x . By the chain rule, if $y(x)$ is any solution to the DE on some interval, then

$$\frac{d}{dx} G(y(x)) = G'(y(x))y'(x) = g(y(x))y'(x) = f(x) = \frac{d}{dx} F(x) .$$

Thus

$$\begin{aligned} G(y(x)) &= F(x) + C . \\ G(y) &= F(x) + C \end{aligned}$$

Expresses y implicitly as a function of x . You may be able to use algebra to solve this equation explicitly for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

Solution (differential magic): Treat $\frac{dy}{dx}$ as a quotient of differentials dy, dx , and multiply and divide the DE to "separate" the variables:

$$\begin{aligned} \frac{dy}{dx} &= \frac{f(x)}{g(y)} \\ g(y)dy &= f(x)dx . \end{aligned}$$

Antidifferentiate each side with respect to its variable (!)

$$\int g(y)dy = \int f(x)dx , \text{ i.e.} \\ G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C . \quad \text{Agrees!}$$

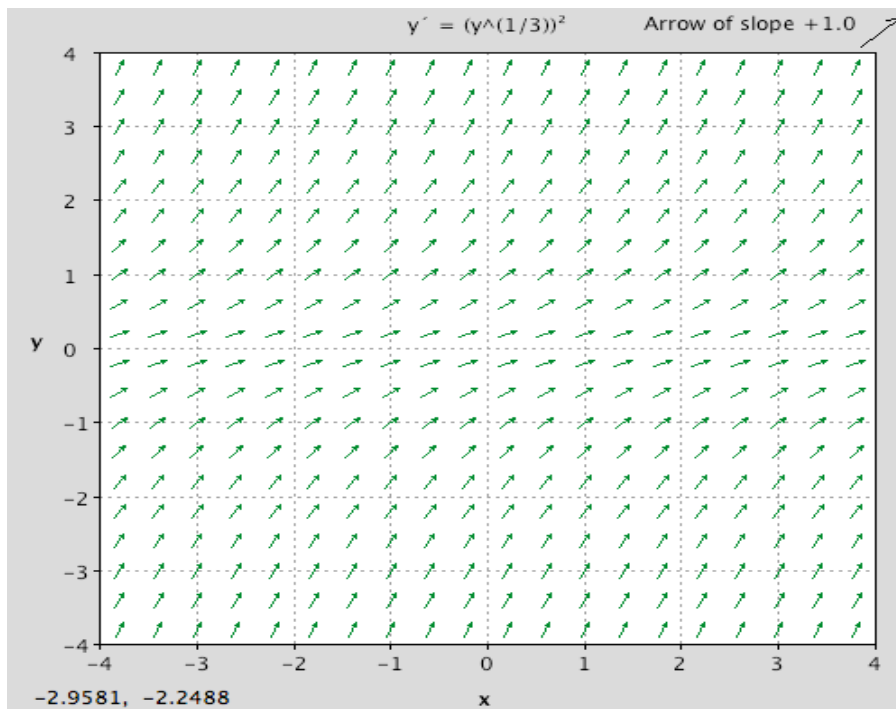
Exercise 3a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$

$$y(0) = 0$$

3b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) This is related to what happened in Exercise 1, only worse.

3c) Sketch some of these singular solutions onto the slope field below.



Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)

Existence - uniqueness theorem for the initial value problem

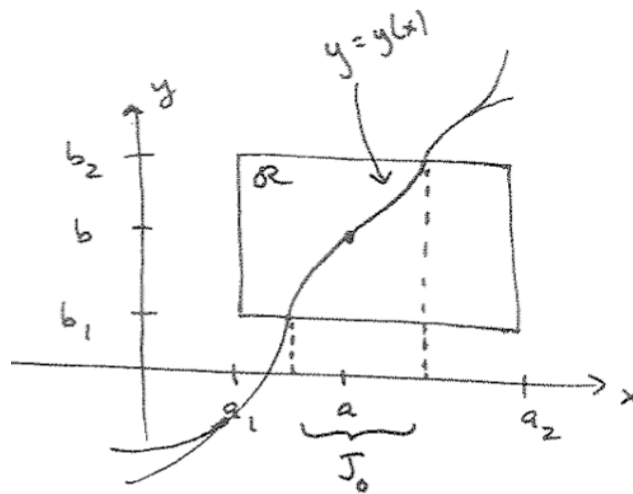
Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$

$$y(a) = b$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the x - y plane.
- Existence: If $f(x, y)$ is continuous in \mathcal{R} (i.e. if two points in \mathcal{R} are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined some subinterval $J \subseteq [a_1, a_2]$.
- Uniqueness: If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in \mathcal{R} , then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition that the y -partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



Exercise 4: Discuss how the existence-uniqueness theorem is consistent with our work in Wednesday's Exercises 2-3, and in today's Exercises 1-3, where we were able to find explicit solution formulas.