Math 2250-1
Wed August 22
Finish 1.2. Differential equations of the form $y'(x) = f(x)$.
Then begin 1.3: slope fields and the geometric interpretation of first order IVP's.

- We still have two exercises to finish from Tuesday's notes: the skidding car problem and the swimmer problem. There are lots of important ideas that will come up as we discuss the mathematical modeling and solutions of these two particular problems.

Exercise 1. Reconstructing the position function from a graph of the velocity function: A car is traveling along a marked road which we may consider equivalent to an x-axis. Set $x(0) = x_0 = 0$. Recorded below is a graph of the car velocity $v(t)$ in miles per minute.
a) Find a formula for $x(t)$, for the interval $0 \le t \le 3$ minutes, and graph $x = x(t)$.
b) Use the formula for $x(t)$ to determine how far the car traveled in those three minutes.
c) The distance the car traveled is also computable in terms of an area associated to the velocity graph. The reason has to do with the Fundamental Theorem of Calculus. Explain the connection, and verify that this area agrees with your answer from (b).
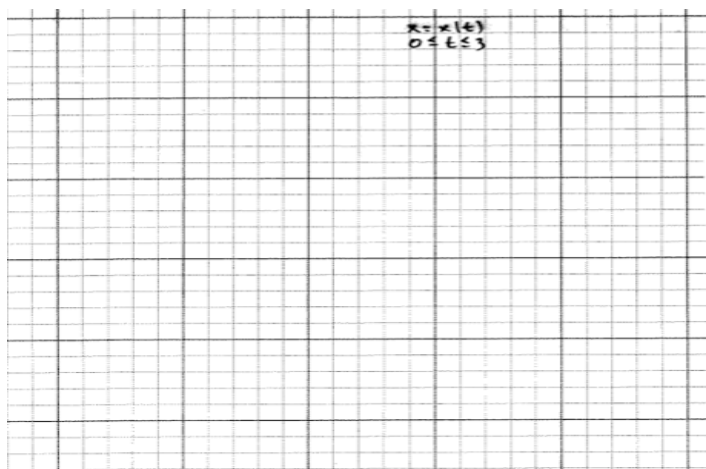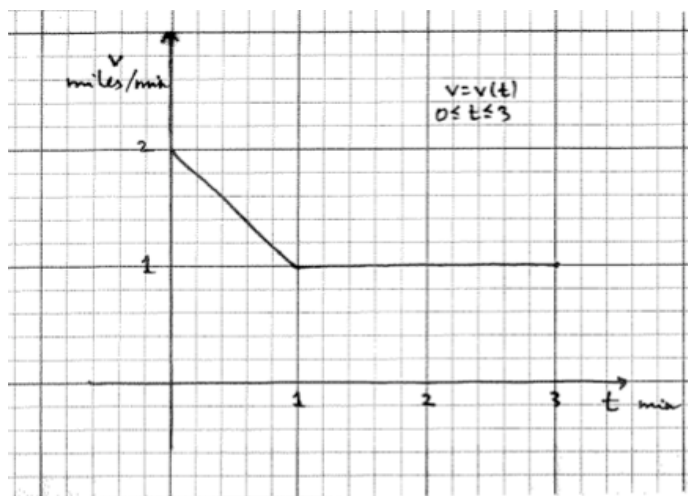
Consider the first order DE IVP for a function $y(x)$:
$$y' = f(x, y) \quad , y(x_0) = y_0 \,.$$

If $y(x)$ is a solution to this IVP and if we consider its graph $y = y(x)$, then the IC means the graph must pass through the point (x_0, y_0). The DE means that at every point (x, y) on the graph the slope of the graph must be $f(x, y)$. This gives an alternate way of understanding the graph of the solution $y(x)$ even without ever actually finding a formula for $y(x)$!

Consider a **slope field** near the point (x_0, y_0): at each nearby point (x, y), assign the slope given by $f(x, y)$. You can represent this in a picture by using small line segments placed at representative points (x, y), having slopes $f(x, y)$. An analogous concept is a **vector field**, which assigns vectors to points. You've possibly discussed that notion in the context of electric, magnetic, or gravitional force fields, for physics and math applications. In the present case we're using fields of slopes rather than vectors.

<u>Excercise2</u>: Consider the differential equation $\dfrac{dy}{dx} = x - 3$, and then the IVP with $y(1) = 2$.

a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \le x \le 5, 0 \le y \le 6$. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line. (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.
b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.
c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).
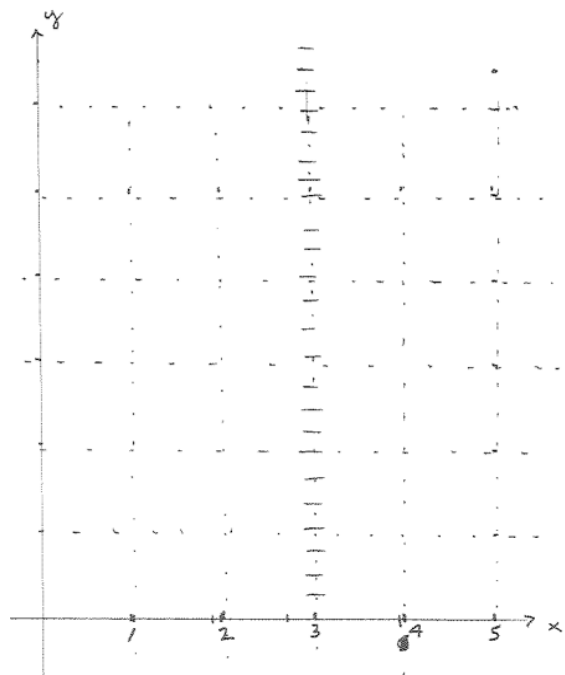
The procedure of drawing the slope field $f(x,y)$ associated to the differential equation $y'(x) = f(x,y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = f(x,y)$$

one can make discrete steps in x and y, starting from the initial point (x_0, y_0). In this way one can approximate solution functions to IVPs, and their graphs. You can find an applet to do this by googling "dfield" (stands for "direction field", which is a synonym for slope field). Here's a picture like the one we sketched by hand on the previous page. The solution graph was approximated using numerical ideas as above, and this numerical technique works for much more complicated differential equations, e.g. when solutions exist but don't have closed form formulas.
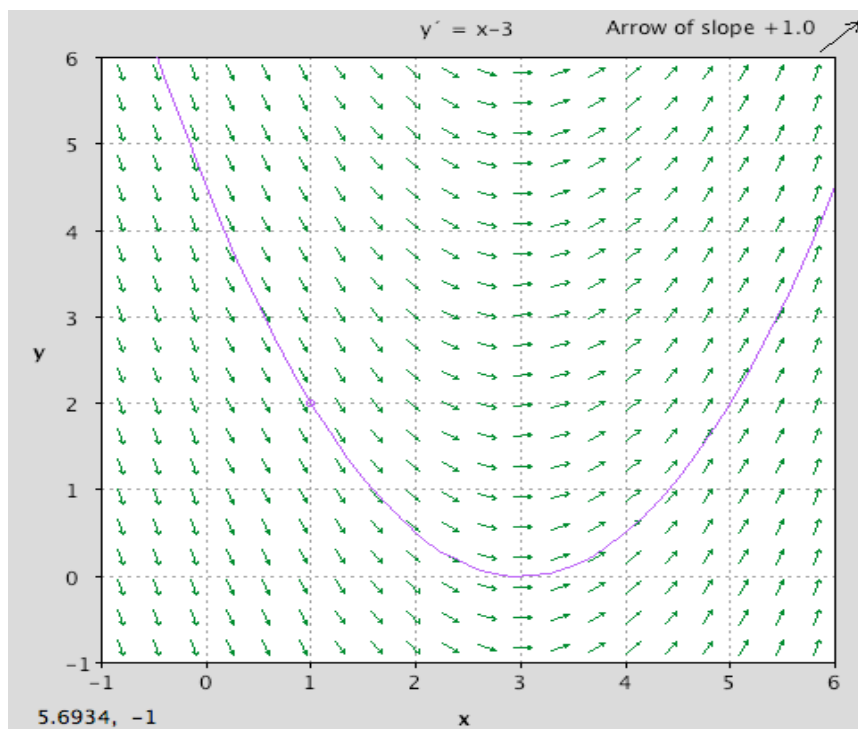
Exercise 3: Consider the IVP

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

a) Check that $y(x) = x + 1 + C e^x$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

b) Solve the IVP by choosing appropriate C.

c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

d) Compare your work in (c) with the picture created by dfield on the next page.....If there's time we'll go to the dfield URL to see how the picture was created...you'll be doing this in next week's homework assignment.

y´ = y−x   Arrow of slope +1.0

1.8478, −2.5837