Math 2250-1, Tuesday August 21

Recall that all course info - syllabus, class notes, homework, etc. is posted at our web page
http://www.math.utah.edu/~korevaar/2250fall12
Let's look quickly at the first homework assignment and its topics. Note that the text often indicates the key concepts as headers of homework subsections.

• Recall from Monday that a 1^{st} order DE is an equation involving a function and its first derivative. We may chose to write the function and variable as $y = y(x)$. In this case the differential equation as an equation equivalent to one of the form

$$F(x, y, y') = 0 .$$

For example

$$x\,y + (y')^2 = 3$$

is a first order differential equation, because it is equivalent to $x\,y + (y')^2 - 3 = 0$.

• We can often use algebra to solve for y', to get what we call the **standard form** for a first order DE:

$$y' = f(x, y) .$$

• If we want our solution function to a DE to also satisfy $y(x_0) = y_0$, and if our DE is written in standard form, then we say that we are solving an **initial value problem** (IVP):

$$y' = f(x, y)$$
$$y(x_0) = y_0 .$$

<u>Exercise 1</u>: Suppose we wanted to find a solution $y(x)$ to the differential equation

$$x\,y + (y')^2 = 3$$

so that $y(1) = -1$. Write the corresponding initial value problem, using the standard form for the differential equation.

Finish Exercises 3 and 4 from Monday:

Monday Exercise 3: The mathematical model in which the rate of change of a population $P(t)$ is proportional to that population is expressed mathematically as

$$\frac{dP}{dt} = k\,P$$

where k is the proportionality constant.

3a) Find all solutions to this differential equation by using the chain rule backwards.

We solved the problem as follows. We rewrote the DE as

$$\frac{P'(t)}{P(t)} = k\,.$$

One of us :-) was able to use the chain rule in reverse to notice that the left side can be rewritten as

$$\frac{d}{dt}\ln(P(t)) = k$$

so

$$\ln(P(t)) = \int k\,dt = k\,t + C_1\,.$$

exponentiating both sides yields

$$e^{\ln(P(t))} = e^{C_1} e^{k\,t}$$

$$P(t) = C\,e^{k\,t} \qquad (C = e^{C_1})$$

Since in this formula $P(0) = C\,e^0 = C$ we often write the solution as

$$P(t) = P_0\,e^{k\,t}$$

where P_0 is the constant $C = P(0)$.

3b) The method of "separation of variables" is taught in most Calc I courses, and we'll cover it in detail in section 1.3. It's an algorithm which hides the "chain rule in reverse" technique by treating the derivative $\frac{dP}{dt}$ as a quotient of differentials. Recall this magic algorithm to recover the solutions from (3a).

Monday Exercise 4) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate proportional to to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use this model to derive the differential equation

$$\frac{dT}{dt} = -k(T - A).$$

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead?

c) Use this model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is $70\,^\circ$F. An hour later the body temperature has decreased to $60\,^\circ$. It's been a winter inversion in SLC, with constant ambient temperature $30\,^\circ$. Assuming the Newton's law model, estimate the time of death.

Section 1.2: differential equations equivalent to ones of the form $y'(x) = f(x)$, which we solve by direct antidifferentiation.

An important class of such problems are velocity/acceleration problems from physics. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t).$$

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

Exercise 2:
a) If the units for position are meters m and the units for time are seconds s, what are the units for velocity and acceleration? (These are *mks* units.)
b) Same question, if we use the English system in which length is measure in feet and time in seconds. Could you convert between *mks* units and English units?

Exercise 3:
Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a.$$

(This could happen for vertical motion, e.g. near the earth's surface with $a = g \approx 9.8 \frac{m}{s^2} \approx 32 \frac{ft}{s^2}$, as well as in other situations.)
a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.
b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{max} = \frac{1}{2}\frac{v_0^2}{a}.$$

(This formula may help with some homework problems, as well as the one below.)

Exercise 4: Car accident reconstruction. A driver skids 210 ft. after applying his brakes. He claims to the investigating officers that he was going 25 miles per hour before trying to stop. A police test of his vehicle shows that if the brakes are applied to force a skid at an initial speed of 25 $\frac{mi}{h}$ then the auto skids only 45 ft. Assuming that the car is decelerating at a constant rate while skidding, about how fast was the driver really going?

Exercise 5: (See text, page 16). A swimmer wishes to cross a river of width $w = 2a$, by swimming directly towards the opposite side, with constant transverse velocity v_S. The river velocity is fastest in the middle, is given by an even function of x, for $-a \le x \le a$, and with velocity equal to zero at the river banks. For example, it could be that

$$v_R(x) = v_0\left(1 - \frac{x^2}{a^2}\right).$$

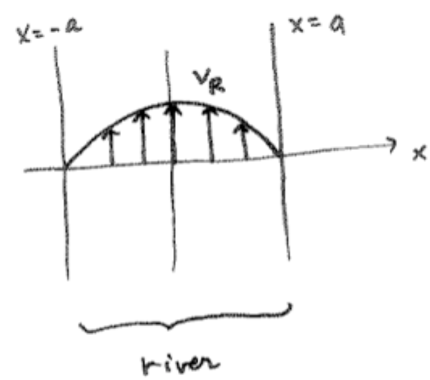
See the configuration sketches below.

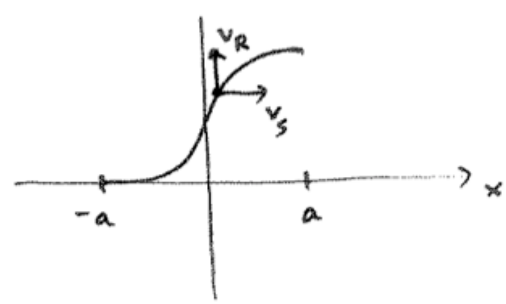
a) If the swimmer location is given by $(x(t), y(t))$ at time t, then translate the information above into expressions for $x'(t)$ and $y'(t)$.

b) The parametric curve describing the swimmer's location can also be expressed as the graph of a function $y = y(x)$. Show that

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right).$$

c) Set up and solve the initial value problem for this DE, to figure out how far downstream the swimmer will be when she reaches the far side of the river.





d) If $v_0 = 9\ \frac{mi}{h}$ and $v_S = 2\ \frac{mi}{h}$, and the river is one mile wide, how far downstream does the swimmer end up?