Name.....Solutions.....

I.D. number.....

Math 2250-1
FINAL EXAM
December 10, 2012

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	_______	20
2	_______	10
3	_______	20
4	_______	15
5	_______	10
6	_______	15
7	_______	20
8	_______	15
9	_______	25
total	_______	150

1) Consider a boat which starts at rest at time $t = 0$ sec, is accelerated in a straight path by an engine that provides a constant 800 N of force, and which is also subject to drag forces of 20 N for each $\frac{m}{s}$ of velocity. The boat has mass 400 kg.

1a) Use your modeling ability to show that the boat velocity $v(t)$ (in meters per second) satisfies the initial value problem

$$v'(t) = 2 - .05\,v$$
$$v(0) = 0$$

(5 points)

Newton 2nd: $mx'' = \text{net forces}$

$$400v' = 800 - 20v$$

$\div 400$ IVP $\begin{cases} v' = 2 - \frac{20}{400}v = 2 - \frac{1}{20}v = 2 - .05v \\ v(0) = 0 \end{cases}$

1b) Solve the initial value problem in 1a.

(10 points)

$$v' + .05v = 2$$
$$e^{.05t}(v' + .05v) = 2e^{.05t}$$
$$e^{.05t}v = \int 2e^{.05t}\,dt = \frac{2}{.05}e^{.05t} + C = 40e^{.05t} + C$$

$\div e^{.05t}$: $v = 40 + Ce^{-.05t}$

$$v(0) = 0 = 40 + C \Rightarrow C = -40$$
$$\boxed{v = 40 - 40e^{-.05t}}$$

1c) How far does the boat travel in the first 20 seconds?

(5 points)

$$x(20) - x(0) = \int_0^{20} x'(t)\,dt = \int_0^{20} 40 - 40e^{-.05t}\,dt$$
$$= 40t + 800e^{-.05t}\Big]_0^{20}$$
$$= 800 + 800e^{-1} - 800$$
$$\boxed{= \frac{800}{e}\ m}$$

2) Here is a matrix and its reduced row echelon form:

$$A := \left[\begin{array}{cccc|c} 3 & 9 & -5 & 1 & 0 \\ -2 & -6 & 2 & -2 & 0 \\ 1 & 3 & -2 & 0 & 0 \end{array}\right] \quad \text{reduced row echelon form of A:} \quad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2a) Find the general solution to the homogeneous matrix equation $A\underline{x} = \underline{0}$. Write your solution in linear combination form.

(6 points)

$x_4 = t$
$x_3 = -t$
$x_2 = p$
$x_1 = -3p - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3p-2t \\ p \\ -t \\ t \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + p\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad t, p \in \mathbb{R}.}$$

2b) Let $\underline{b}$ be the first column of A, i.e.

$$\underline{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

What is the general solution to the non-homogeneous equation $A\,\underline{x} = \underline{b}$? Hint: since you already know the homogeneous solution from part a, you only need to find a particular solution and superposition in order to deduce your answer. Since a matrix times a vector is just a linear combination of the columns, and since you want $A\,\underline{x}_p$ to be the first column of A, there is a natural choice for $\underline{x}_p$.

(4 points)

$$\begin{bmatrix} 3 & 9 & -5 & 1 \\ -2 & -6 & 2 & -2 \\ 1 & 3 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \qquad \text{so } \vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

general soln

$$\boxed{\vec{x} = \vec{x}_p + \vec{x}_H = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + p\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad t, p \in \mathbb{R}}$$

3) Consider the following initial value problem, which could arise from Newton's second law in a forced mass-spring oscillation problem:

$$x''(t) + 6\,x'(t) + 25\,x(t) = 50$$
$$x(0) = 0$$
$$x'(0) = 6\,.$$

3a) If the applied force in this problem is $100\,N$, then what are the corresponding values for the Hooke's constant, mass, and damping coefficients? Include correct units.

(5 points)

$$m x'' + c x' + k x = F = 100$$

match

$$\Rightarrow \quad \frac{m}{2} x'' + \frac{c}{2} x' + \frac{k}{2} x = 50$$

so $\frac{m}{2} = 1 \Rightarrow m = 2\ kg$; $\frac{c}{2} = 6 \Rightarrow c = 12\ \frac{Ns}{m} = 12\ \frac{kg}{s}$; $\frac{k}{2} = 25 \Rightarrow$
$k = 50\ N/m$

3b) Solve this initial problem using the methods of Chapter 5, based on particular and homogeneous solutions.

(15 points)

x_H: $x'' + 6x' + 25x = 0$

$p(r) = r^2 + 6r + 25 = (r+3)^2 + 16$

roots $(r+3)^2 = -16 \Rightarrow r+3 = \pm 4i \Rightarrow r = -3 \pm 4i$

$\Rightarrow \quad x_H(t) = c_1 e^{-3t}\cos 4t + c_2 e^{-3t}\sin 4t$

x_p: try $x_p = C \Rightarrow x_p'' + 6x_p' + 25x_p = 0 + 0 + 25C \overset{\text{want}}{=} 50$

$\Rightarrow C = 2.$

$\Rightarrow x_p = 2.$

$\Rightarrow \quad x(t) = x_p + x_H = 2 + c_1 e^{-3t}\cos 4t + c_2 e^{-3t}\sin 4t$

$x(0) = 0 = 2 + c_1 \Rightarrow c_1 = -2.$

$x'(0) = 6 = 0 - 3c_1 + 4c_2 = 6 + 4c_2 \Rightarrow c_2 = 0$

$$\Rightarrow \quad \boxed{x(t) = 2 - 2e^{-3t}\cos 4t}$$

4) Re-solve the initial value problem in 3 using Laplace transforms:

$$x''(t) + 6x'(t) + 25x(t) = 50$$
$$x(0) = 0$$
$$x'(0) = 6 .$$

(15 points)

$\mathcal{L}$: $\quad s^2 X(s) - 0s - 6 + 6\left(sX(s) - 0\right) + 25X(s) = \frac{50}{s}$

$$X(s)\left(s^2+6s+25\right) = \frac{50}{s} + 6 = \frac{50+6s}{s}$$

$$X(s) = \frac{50+6s}{s\left(s^2+6s+25\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+6s+25}$$

$$50+6s = A\left(s^2+6s+25\right) + (Bs+C)s$$

@ $s=0$: $\quad 50 = 25A \Rightarrow A = 2$

$$50+6s = 2\left(s^2+6s+25\right) + (Bs+C)s$$
$$= s^2(2+B) + s(12+C) + 1(50)$$

$$\Rightarrow \begin{array}{ll} 0 = 2+B & \Rightarrow B = -2 \\ 6 = 12+C & \quad C = -6. \end{array}$$

$$X(s) = \frac{2}{s} + \frac{-2s-6}{s^2+6s+25}$$
$$= \frac{2}{s} - 2\frac{(s+3)}{(s+3)^2+16}$$

$\mathcal{L}^{-1}$: $\Rightarrow \quad x(t) = 2 - 2e^{-3t}\cos 4t$

5a) Use Laplace transforms to solve the initial value problem

$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

(7 points)

$\mathcal{L}$: $s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m} \frac{s}{s^2+\omega_0^2}$

$$X(s)(s^2+\omega_0^2) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2} + s x_0 + v_0$$

$$X(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2} + \frac{s x_0}{s^2+\omega_0^2} + \frac{v_0}{s^2+\omega_0^2}$$

using table

$$\boxed{x(t) = \frac{F_0}{m}\frac{t}{2\omega_0}\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

5b) What physical phenomenon is exhibited by solutions to 5a? Explain.

(3 points)

resonance

pure resonance
has a term with "time varying amplitude" that grows linearly.

(It occurs in undamped oscillation problems when the system is forced at its natural frequency)

6a) Find the general solution to the first order system of differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(10 points)

Hint: The eigenvalues of the matrix are negative integers.

$$p(\lambda) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda+3)(\lambda+2)-2 = \lambda^2+5\lambda+4 = (\lambda+4)(\lambda+1)$$

roots $\lambda = -1, -4$.

$\lambda = -1$; $(A+I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$e^{\lambda t}\vec{v} = e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -4$; $(A+4I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

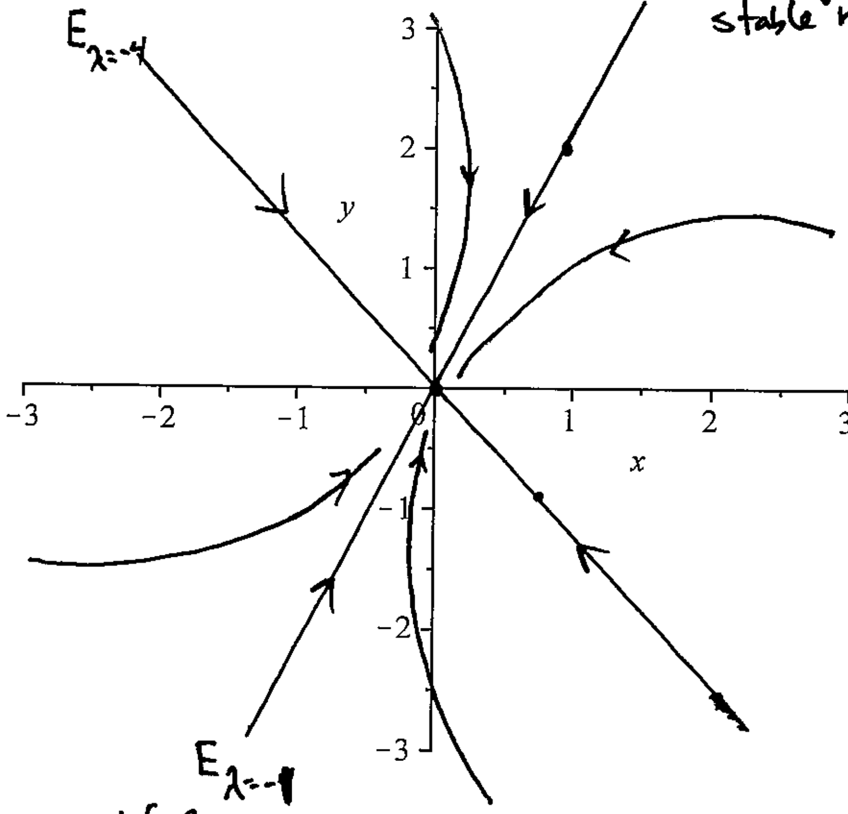
$e^{\lambda t}\vec{v} = e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

6b) Sketch the phase portrait for this linear homogeneous first order system, based on your work and eigendata in part a. Classify the equilibrium point at the origin. Since $\lambda_1, \lambda_2 < 0$

(5 points)

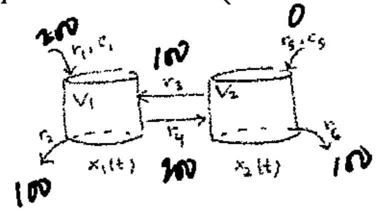
$\vec{0}$ is an asymptotically stable nodal sink.



$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\uparrow$
$\to 0$ faster as $t \to \infty$, so sol'ns approach origin along E_{-1} eigenspace. But $\to \infty$ faster as $t \to -\infty$, so sol'ns come from this direction

7) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t)$, $x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



7a) Suppose $r_2 = r_3 = r_6 = 100$, $r_1 = r_4 = 200$, $r_5 = 0\ \frac{gal}{hour}$. Explain why the volumes $V_1(t)$, $V_2(t)$ remain constant.

(4 points)

$V_1'(t) = 200 + 100 - 100 - 200 = 0$ so $V_1 = \text{const}$

$V_2'(t) = 200 + 0 - 100 - 100 = 0$ so $V_2 = \text{const}.$

7b) Using the flow rates above, $c_1 = 0.6$, $c_5 = 0\ \frac{lb}{gal}$, $V_1 = V_2 = 100\ gal$, show that the amounts of solute $x_1(t)$ in tank 1 and $x_2(t)$ in tank 2 satisfy

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix}.$$

(6 points)

$$x_1' = r_i c_i - r_o c_o$$
$$= 200 \cdot (.6) + 100 \frac{x_2}{V_2} - 300 \frac{x_1}{V_1} = 120 + x_2 - 3x_1$$

$$x_2' = r_i c_i - r_o c_o$$
$$= 200 \frac{x_1}{V_1} - 200 \frac{x_2}{V_2} = 2x_1 - 2x_2$$

So,

$$x_1' = -3x_1 + 2x_2 + 120$$
$$x_2' = 2x_1 - 2x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix} \checkmark$$

7c) Solve the initial value problem for the system

$x_1(0) = x_2(0) = 0$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix},$$

assuming there is initially no solute in either tank. Hint: Find a particular solution which is a constant vector, and then use $\underline{x} = \underline{x}_P + \underline{x}_H$ to solve the IVP. Notice that you have already found $\underline{x}_H$ in problem 6.

(10 points)

try $\vec{x}_p = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$; $\vec{x}_p' = A\vec{x}_p + \vec{b}$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 0 &= -3c_1 + c_2 + 120 \\ 0 &= 2c_1 - 2c_2 \end{aligned}$$

$$\begin{aligned} 3c_1 - c_2 &= 120 \\ c_1 - c_2 &= 0 \end{aligned}$$

$E_1 - E_2 \Rightarrow 2c_1 = 120 \Rightarrow c_1 = 60 \Rightarrow c_2 = 60$

$$\vec{x}_p = \begin{bmatrix} 60 \\ 60 \end{bmatrix}.$$

so, from 6,

$$\vec{x} = \vec{x}_p + \vec{x}_H = \begin{bmatrix} 60 \\ 60 \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

@ $t=0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{aligned} 0 &= 60 + c_1 + c_2 \\ 0 &= 60 + 2c_1 - c_2 \end{aligned}$$

add eqtns $\Rightarrow 0 = 120 + 3c_1 \Rightarrow c_1 = -40$

$\Rightarrow c_2 = -20$

$$\Rightarrow \boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \end{bmatrix} - 40e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 20e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

i.e.

$$\begin{aligned} x_1(t) &= 60 - 40e^{-t} - 20e^{-4t} \\ x_2(t) &= 60 - 80e^{-t} + 20e^{-4t} \end{aligned}$$

8a) Find the general solution to this second order system of differential equations, which could arise when modeling a coupled mass-spring system. Hint: You have already computed eigendata for the relevant matrix in problem 6.

$$x_1''(t) = -3x_1 + x_2$$
$$x_2''(t) = 2x_1 - 2x_2 .$$

(6 points)

from 6, for

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$$

$\lambda = -1$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\omega = \sqrt{1} = 1$

$\lambda = -4$, $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\omega = \sqrt{4} = 2$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{(c_1 \cos t + c_2 \sin t)}_{C_1 \cos(t-\alpha_1)} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underbrace{(c_3 \cos 2t + c_4 \sin 2t)}_{C_2 \cos(2t-\alpha_2)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

8b) Describe the two fundamental modes of vibration for this spring system.

(4 points)

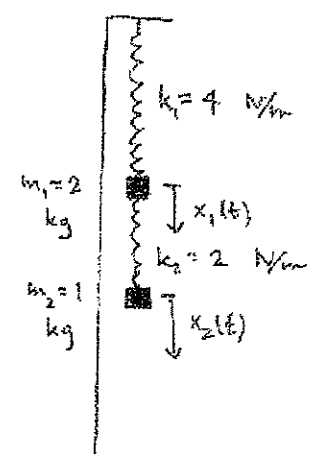
- slow mode ($c_3 = c_4 = 0$): masses in phase amplitude of 2nd mass simple harmonic motion is twice that of 1st mass, $\omega = 1$
- fast mode ($c_1 = c_2 = 0$): masses out of phase, equal amplitude, $\omega = 2$

8c) For Hooke's constants and masses as shown below, show that the displacement functions $x_1(t), x_2(t)$ of the two masses below (from their equilibrium hanging positions) satisfy the system of differential equations in this problem.

(5 points)

mass 1:

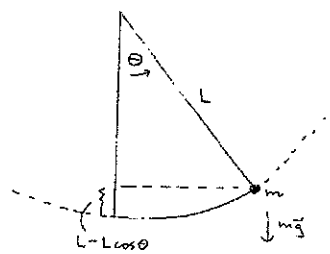
$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$2x_1'' = -4x_1 + 2(x_2 - x_1) = -6x_1 + 2x_2$$
$$\div 2 \quad x_1'' = -3x_1 + x_2$$



mass 2:

$$m_2 x_2'' = -k_2(x_2 - x_1)$$
$$1 x_2'' = -2(x_2 - x_1)$$
$$x_2'' = 2x_1 - 2x_2$$

9) We have studied and returned to the rigid rod pendulum several times in this course. This is the freely-rotating configuration indicated in the diagram below:



Using conservation of energy we have derived the autonomous second order differential equation that describes the angle $\theta(t)$ of the mass from the vertical reference line, at time t, arriving at

$$(1) \qquad \theta''(t) + \frac{g}{L}\sin(\theta(t)) = 0\,.$$

9a) That second order DE above is equivalent to the autonomous first order system of two differential equations

$$(2) \qquad \begin{aligned} x'(t) &= y \\ y'(t) &= -\frac{g}{L}\sin(x)\,. \end{aligned}$$

Explain this equivalence.

(4 points)

Let $\Theta(t)$ solve (1). Then define $x(t) = \Theta(t)$
$y(t) := \Theta'(t)$

It follows that $\left.\begin{aligned} x' &= y \\ y' &= \Theta'' = -\frac{g}{L}\sin\theta = -\frac{g}{L}\sin x \end{aligned}\right\}$ (2)

Conversely, if $x(t), y(t)$ solve (2),
define $\Theta(t) = x(t)$. Then $\Theta'(t) = y$ so $y'(t) = \Theta'' = -\frac{g}{L}\sin x = -\frac{g}{L}\sin\Theta$
so $\Theta'' + \frac{g}{L}\sin\Theta = 0$ (2).

9b) Find all equilibrium (i.e. constant) solutions to the first order system of DE's above. Explain how these equilibrium solutions are related to the constant solutions of the second order differential equation for rigid-rod pendulum.

(6 points)

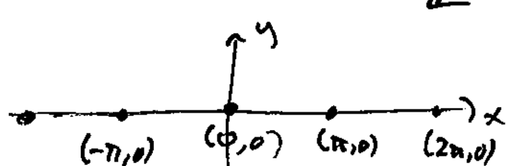
$$\begin{aligned} x' &= y & &= F \\ y' &= -\frac{g}{L}\sin x & &= G. \end{aligned}$$

(x_*, y_*) equil $\Leftrightarrow$ $F(x_*, y_*) = G(x_*, y_*) = 0$

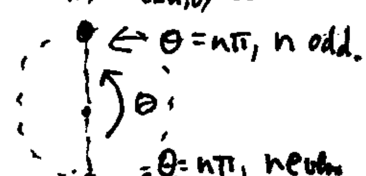
$\Rightarrow y = 0$

$\Rightarrow \sin x = 0 \Rightarrow x = n\pi,\ n \in \mathbb{Z}$.

So equil soltns are $(0, n\pi)$, $n \in \mathbb{Z}$



- when n is even
  $n\pi$ is a multiple of 2π
  for equil soltn is mass at rest at <u>bottom</u> of rotating pendulum
- when n is odd ~~pen~~ mass is at rest at top of circle.



system repeated for your convenience:

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x)\ .$$

9c) Use linearization and the Jacobian matrix to classify the equilbrium solutions to the first order system above. Indicate what you can deduce about the stability of these equilibrium solutions based only on the linearization.

(10 points)

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & 0 \end{bmatrix}.$$

- if $x = n\pi$, n even $\Rightarrow J(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$ $\quad |J-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{L}$

roots $\lambda^2 + \frac{g}{L} = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{g}{L}}$

$\lambda =$ purely imag $\Rightarrow$ $(n\pi, 0)$ n even is stable center for linearization, but indeterminate for non-linear sys

- if $x = n\pi$, n odd

$\Rightarrow J(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix}$

$|J-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ g/L & -\lambda \end{vmatrix} = \lambda^2 - \frac{g}{L} = 0 \Rightarrow \lambda^2 = \frac{g}{L} \Rightarrow \lambda = \pm\sqrt{\frac{g}{L}}$ (unstable) saddle

9d) Use the separation of variables method we discussed in class (or if you prefer, use conservation of energy), to show that the parametric solution curves $(x(t), y(t))$ to the first order system above lie on the level curves for a certain function. Explain why this shows that the equilibrium solutions that are borderline based on the linearization work in 9c, are actually stable for the original non-linear system.

(5 points)

for soltn trajectory $(x(t), y(t))$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-\frac{g}{L}\sin x}{y}$$

is separable, so soltns follow level curves of some fcn:

$$y\,dy = -\frac{g}{L}\sin x\, dx$$

integrate: $\frac{1}{2}y^2 = \frac{g}{L}\cos x + C$

$$\frac{1}{2}y^2 - \frac{g}{L}\cos x = C.$$

So ~~curve~~ solution curves are level curves of $E(x,y) = \frac{1}{2}y^2 - \frac{g}{L}\cos x$.
(in fact, $E(x,y)$ is a constant times the KE + PE for the moving mass.).

$$E(x,y) = \frac{1}{2}y^2 - \frac{g}{L}\cos x = \underbrace{f(y)}_{\substack{\text{global min} \\ @\, y=0}} + \underbrace{g(x)}_{\substack{\text{global min} \\ @\, x=n\pi,\ n \text{ even}}}$$

Thus $E(x,y)$ has global minimum value at pts $(n\pi, 0)$, n even, so these equilibrium sol'ns are stable. (Soltns starting nearby have almost this min energy, so must stay close.

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\frac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt\cos kt)$	$\frac{s^2}{(s^2+k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\frac{1}{s}\tanh\frac{as}{2}$
t	$\frac{1}{s^2}$	$\left[\!\left[\frac{t}{a}\right]\!\right]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		