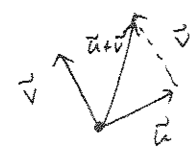
Math 2250-1
Friday 9/30

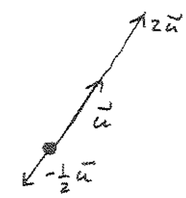
Chapter 4!
will use the matrix theory from chapter 3 to understand concepts related to

<u>linear</u> <u>combinations</u> of <u>vectors</u> (and functions)

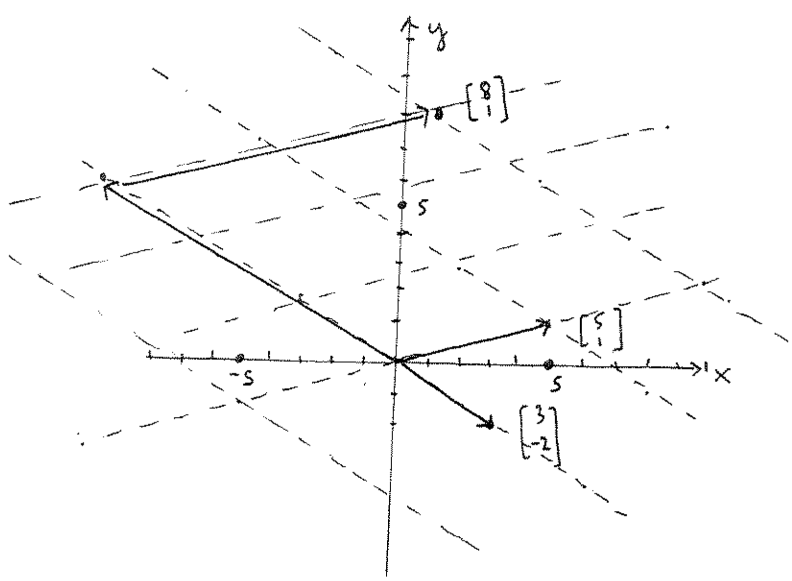
A linear combination of a collection of vectors (or functions) is a sum of multiples of those vectors.

<u>Exercise 1</u> (to review geometric meaning of vector addition & scalar multiplication in $\mathbb{R}^n$, and to interpret linear combination problems geometrically.)





Can you get to $\begin{bmatrix}1\\8\end{bmatrix}$ in $\mathbb{R}^2$ from the origin, moving only in the $\begin{bmatrix}3\\-2\end{bmatrix}$ and $\begin{bmatrix}5\\1\end{bmatrix}$ directions?



We seek linear combo coefficients s, t, so

$$s\begin{bmatrix}3\\-2\end{bmatrix}+t\begin{bmatrix}5\\1\end{bmatrix}=\begin{bmatrix}1\\8\end{bmatrix}$$

1a) Find s & t!

1b) Can we get to any point $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$, starting from the origin and moving only in directions parallel to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$?

i.e. does $s\begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ always have a solution $\begin{bmatrix} s \\ t \end{bmatrix}$?

Is $\begin{bmatrix} s \\ t \end{bmatrix}$ unique?

Exercise 2a) Show that linear combinations of the functions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-x}$ solve the differential equation

$$y'' - 2y' - 3y = 0$$

2b) solve for A, B so that

$$Ae^{3x} + Be^{-x}$$

solves the IVP $\begin{array}{l} y(0) = 1 \\ y'(0) = -5 \end{array}$

2c) Given the fact that solutions to the IVP $\begin{cases} y''-2y'-3y=0 \\ y(0)=b_1 \\ y'(0)=b_2 \end{cases}$

are unique, is every sol'tn to ~~the~~ such an IVP
a linear combination of e^{3x} and e^{-x}?
Are the linear combination coeff's A, B for such a solution unique?

the language we will use:

If $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is a collection of vectors (say in $\mathbb{R}^m$) [this language will also apply to functions]
then $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ means

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$$

for some choice of the scalar (numbers) $c_1, c_2, \dots c_k$. These numbers are called the linear combination coefficients or coordinates

the span of $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is the collection of all $\vec{w}$'s which are linear combinations of $\vec{v}_1, \dots \vec{v}_k$.

connection to matrix multiplication:

$$\underset{\uparrow \atop \vec{v}_1}{c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}} + \underset{\uparrow \atop \vec{v}_2}{c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}} + \dots + c_k\begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} = \begin{bmatrix} c_1a_{11} + c_2a_{12} + \dots + c_ka_{1k} \\ c_1a_{21} + \dots + c_ka_{2k} \\ \vdots \\ c_1a_{m1} + \dots + c_ka_{mk} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & & a_{2k} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & & a_{mk} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

i.e. $A\vec{c} = c_1 col_1(A) + c_2 col_2(A) + \dots + c_k col_k(A)$ $\quad A_{m \times k}$.
is a linear combination of the colums of A

Exercise 3 Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}$ be vectors in $\mathbb{R}^3$

3a) Is $\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?
(Use Maple output).

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}.$$

3b) Are the linear combo coeff's c_1, c_2, c_3 unique?

3c) Explain why

$$\text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$

What geometric object is this span?

Hint:
$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & x \\ 1 & 1 & -1 & y \\ 1 & -5 & 11 & z \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & y \\ 0 & 3 & -6 & -x+2y \\ 0 & 0 & 0 & -2x+3y+z \end{array}\right]$$

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> with(LinearAlgebra) :
  AT := Matrix(3, 3, [2, 1, 1, -1, 1, -5, 4, -1, 11]) :
  A := Transpose(AT);
  b := Vector([6, 3, 3]) :
  Aaugb := ⟨A|b⟩;
  ReducedRowEchelonForm(Aaugb);
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$$A := \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & -1 \\ 1 & -5 & 11 \end{bmatrix}$$

$$Aaugb := \begin{bmatrix} 2 & -1 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 1 & -5 & 11 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 4 a) If you have 3 vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ in $\mathbb{R}^3$, what matrix condition is equivalent to: $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$ <u>and</u> for each $\vec{b}$ in $\mathbb{R}^3$ with $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$, the linear combination coef's c_1, c_2, c_3 are unique?

(In this case we call $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a <u>basis</u> for $\mathbb{R}^3$)

4b) Why does every basis of $\mathbb{R}^3$ have exactly 3 vectors?

to be continued...