Math 2250-1
§3.6 Determinants
Tues 9/27

A chance to go through spring 2011 midterm tonight 6-7:30 p.m. room TBA.
(you could also ask abt hw due Wed.)

recall: cofactor expansions for det
det and elementary row ops.

DO $\longrightarrow$ $|A| \neq 0$ iff $rref(A) = I$ iff $A^{-1}$ exists.
(Monday notes)

another magic formula

$$|AB| = |A||B|$$

(see Monday notes... not true that $|A+B| = |A|+|B|$).

today: magic formulas for $A^{-1}$ when $|A| \neq 0$.

need: transpose of a matrix (doesn't have to be a square matrix.)

Def If $B_{m\times n} = [b_{ij}]$ then the transpose of B, $B^T$ is defined by

$$entry_{ij}(B^T) := entry_{ji}(B) = b_{ji}$$

the effect of this is to switch rows & columns, since e.g.

$$entry_{ij}(B^T) = entry_j(row_i(B^T))$$
$$\parallel$$
$$entry_{ji}(B) = entry_j(col_i(B))$$

example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

<u>Theorem</u> Recall the <u>cofactor</u> $C_{ij} = (-1)^{i+j} M_{ij}$

where the <u>minor</u> $M_{ij}$ is the det of the $(n-1)\times(n-1)$ matrix obtained by deleting row $i$ and col $j$ of $A$

the <u>adjoint</u> matrix is defined to be the <u>transpose</u> of the cofactor matrix, $[C_{ij}] = \text{cof}(A)$.

i.e.

$$\text{Adj}(A) := \text{cof}(A)^T$$

then, when $A^{-1}$ exists, it has formula

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$$

<u>exercise 1</u>: Show that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ this theorem reproduces our "magic" formula

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

<u>exercise 2</u> for our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $\text{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$, $|A| = 15$

What is $A^{-1}$?

Check ans!

$$\text{adj}(A) = \begin{bmatrix} \phantom{0} & \phantom{0} & \phantom{0} \\ & & \\ & & \end{bmatrix}$$

exercise 3

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \operatorname{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

3a) the (1,1) entry of

$[A][\operatorname{adj}(A)]$ is $15. = 1\cdot 5 + 2\cdot 2 - 1(-6)$

explain why this is det(A), expanded across first row

3b) the (2,1) entry of $[A][\operatorname{adj}(A)]$ is $0\cdot 5 + 3\cdot 2 + 1(-6) = 0.$

Notice, you used the same cofactors as in 3a).
What matrix (which uses 2 rows of A) is this the determinant of?

3c) the (3,2) entry of $[A][\operatorname{adj}(A)]$ is $2\cdot 0 + (-2)(3) + 1(6) = 0$

What matrix (which uses 2 rows of A) is this the det of?

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If you understand 3a, b, c, completely, you have just realized why

$(A)(\operatorname{adj}(A)) = \det(A)\, I$ for every square matrix,

and so, why $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$ when $\det A \neq 0$.

precisely, $\text{entry}_{ii}\, A(\operatorname{adj}(A)) = \sum_{j=1}^{n} a_{ij} C_{ij} = |A|$

$k \neq i$, $\text{entry}_{ik}\, A(\operatorname{adj}(A)) = \sum_{j=1}^{n} a_{ij} C_{kj} = \begin{vmatrix} \text{row}_1(A) \\ \vdots \\ \text{row}_i(A) \\ \vdots \\ \text{row}_i(A) \\ \text{row}_n(A) \end{vmatrix} \begin{matrix} \\ \\ \leftarrow \text{slot } k \\ \\ \leftarrow \text{slot } i \\ \\ \end{matrix} = 0.$

this proves the $A^{-1}$ formula theorem!

There's a related formula for solving for individual components of $\vec{x}$, when $A\vec{x} = \vec{b}$ has unique solutions:

## Cramer's rule

let $\vec{x}$ solve $A\vec{x} = \vec{b}$ for invertible $A$.

then $$x_k = \frac{\left| C_1 \middle| C_2 \middle| \cdots \middle| b \middle| \cdots C_n \right|}{|A|}$$

numerator is det of matrix obtained from $A$ by replacing ~~row~~ column $k$ by $\vec{b}$.

proof: $$\begin{aligned} x_k &= \text{entry}_k\left(A^{-1}\vec{b}\right) \\ &= \text{entry}_k\left(\frac{1}{|A|}\operatorname{adj}(A)\vec{b}\right) \\ &= \frac{1}{|A|}\overrightarrow{\text{row}}_k(\operatorname{adj}(A))\cdot\vec{b} \\ &= \frac{1}{|A|}\underbrace{\sum_{\ell=1}^{n} C_{\ell k} b_\ell} \end{aligned}$$

this is the expansion of the det in numerator of Cramer's rule, down the $k^{th}$ column! ∎

exercise 4: Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

a) with Cramer's rule

b) with $A^{-1}$, using adjoint formula