Math 2250-1
Wed 9/21

HW for next Wed 9/28
is posted; §3.5-3.6
Exam next Thurs 9/29 covers thru §3.6

§3.4 Matrix algebra, 3.5 Matrix inverses

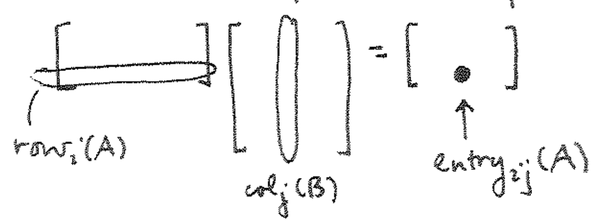
If A is a matrix with m rows, n colums

we write $A_{m \times n}$. We generally write a_{ij} for the row_i, col_j entry of A.
($\text{entry}_{ij}(A)$)

addition If A, B are both $m \times n$
$$\text{entry}_{ij}(A+B) := a_{ij} + b_{ij}$$
scalar multiplication: if $c \in \mathbb{R}$,
$$\text{entry}_{ij}(cA) := c a_{ij}$$

just the same as vector addition & scalar multiplication

$$A_{m \times n} \cdot B_{n \times p} = (AB)_{m \times p}$$



matrix multiplication
$$\text{entry}_{ij}(AB) := \text{row}_i(A) \cdot \text{col}_j(B)$$

(equivalently,
$$\text{col}_j(AB) = \underbrace{A(\text{col}_j(B))}_{\text{matrix times vector}}$$
)

Examples for matrix multiplication:

Rules for this algebra

+ is commutative $\quad A+B=B+A$

+ is associative $\quad (A+B)+C=A+(B+C)$

scalar mult distributes over + $\quad c(A+B)=cA+cB$

mult is associative $\quad A(BC)=(AB)C$

mult distributes over + $\quad A(B+C)=AB+AC$

$$(A+B)C=AC+BC$$

mult <u>not</u> commutative in general : DON'T EXPECT $AB=BA$!

<u>check properties</u> :

matrix algebra continued...

the $(n\times n)$ identity matrix $I := \begin{bmatrix} 1 & & & \\ 0 & 1 & & 0 \\ 0 & 0 & \ddots & \\ & 0 & & 1 \end{bmatrix}$ $\quad$ $entry_{jj}(I) = 1$

$entry_{ij}(I) = 0$ if $i \neq j$

$$col_j(I) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow \text{entry } j \quad := \vec{e}_j$$

$$row_i(I) = [0, 0, \ldots \underset{\substack{\uparrow \\ \text{entry } i}}{1}, 0 \ldots 0] = \vec{e}_i \quad \text{(written as a row vector)}$$

$$A_{m\times n} I_{n\times n} = A$$

check: $entry_{ij}(A\,I) = row_i(A) \cdot col_j(I) = row_i(A) \cdot \vec{e}_j = a_{ij} = entry_{ij}(A)$

$$I_{m\times m} A_{m\times n} = A$$

check:

<u>Definition</u> $A_{n\times n}$ is <u>invertible</u> $:=$ there is a matrix B s.t. $AB = BA = I$

(and then we write A^{-1} for B)

[A can have at most 1 inverse, since if

$AB = BA = I$

and $AC = CA = I$

then $B(AC) = (BA)C$ $\quad$ associative prop

$B\ I = I\ C$

$B = C$!]

<u>Example</u> If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, show $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

What good is A^{-1} ?

Theorem: If A^{-1} exists, then the only solution to $A\vec{x} = \vec{b}$ is
$$\vec{x} = A^{-1}\vec{b}$$

Example continued:

Solve
$$\begin{aligned} x + 2y &= 5 \\ 3x + 4y &= 6 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \xrightarrow{\text{Theorem}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

check ans:

check theorem:

But where did that formula for A^{-1} come from?

Ans: Write $A^{-1} = X$, an unknown matrix

We want
$$AX = I.$$
$$A\left[\mathrm{col}_1(X) \;\vdots\; \mathrm{col}_2(X)\right] = \begin{bmatrix} 1 & \vdots & 0 \\ 0 & \vdots & 1 \end{bmatrix}$$

find both columns at once with rref a double augmented matrix

$$\begin{array}{cc:c:c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \xrightarrow[\text{reduces to}]{\text{show this}} \begin{array}{cc:c:c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}$$

$\uparrow$ $\mathrm{col}_1(X)$ $\nwarrow$ $\mathrm{col}_2(X)$!

Will this always work? Can you find A^{-1} for $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}$?