Name_____
Student I.D._____

Math 2250−1
Quiz 3 Solutions
September 9, 2011

1) Consider the following differential equation:

$$\frac{dx}{dt} = -x^2 + 5\cdot x - 4 .$$

1a) Find the equilibrium solutions. Hint: rewrite the right hand side as $-\left(x^2 - 5\cdot x + 4\right)$ and factor.

(4 points)

Solution*: We factor:*

$$\frac{dx}{dt} = -\left(x^2 - 5\cdot x + 4\right) = -(x-1)\cdot(x-4) .$$

Thus the equilibrium solutions (constant solutions) are $x = 1$ *and* $x = 4$.

1b) Draw a phase diagram, and from this diagram deduce whether your equilibrium solutions are stable or unstable. For stable equilibrium solutions, determine if they are asymptotically stable.

(6 points)

Solution: $\frac{dx}{dt} = -(x-1)\cdot(x-4)$ *satisifies*

$\frac{dx}{dt} < 0$ for $x > 4$ *(negative times positive times positive is negative);* $\frac{dx}{dt} > 0$ for $1 < x < 4$; $\frac{dx}{dt} < 0$ for $x < 1$.

Thus the phase diagram is:

$$\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow 1 \rightarrow \rightarrow \rightarrow \rightarrow 4 \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow .$$

Therefore x=1 is an unstable equilibrium solution and x=4 is an asymptotically stable equilibrium solution.

(Note − could you find the solutions to this DE? You should be able to, even though there isn't time on this quiz for me to ask that question.)

Not graded, but here's how:
Separate:

$$\text{separate: } \frac{dx}{(x-1)\cdot(x-4)} = -dt$$

$$\text{partial fractions: } \frac{1}{3}\cdot\left(\frac{1}{x-4} - \frac{1}{x-1}\right)\cdot dx = -dt$$

$$\Rightarrow \left(\frac{1}{x-4} - \frac{1}{x-1}\right)\cdot dx = -3\cdot dt$$

integrate:

$$\ln\left|\frac{x-4}{x-1}\right| = -3\cdot t + C_1$$

exponentiate: $\left|\frac{x-4}{x-1}\right| = e^{-3\cdot t + C_1}$

$$\Rightarrow \left(\frac{x-4}{x-1}\right) = C\cdot e^{-3\cdot t} = \left(\frac{x_0-4}{x_0-1}\right)\cdot e^{-3\cdot t}.$$

multiply both sides by $(x-1)$ $\Rightarrow$

$$x-4 = C\cdot e^{-3\cdot t}\cdot(x-1)$$

collect terms $\Rightarrow x\left(1-Ce^{-3\cdot t}\right) = 4 - C\cdot e^{-3\cdot t}$

$$\Rightarrow x = \frac{4-C\cdot e^{-3\cdot t}}{1-Ce^{-3\cdot t}}\text{ , with } C = \left(\frac{x_0-4}{x_0-1}\right).$$