Name_______________________

Student I.D.___________________

Math 2250−1
Quiz 12 SOLUTIONS
December 2, 2011

1) Find the general solution $\left[x_1(t), x_2(t)\right]^T$ to the homogeneous system of second order differential equations which could result from a "train" of two masses coupled with a single spring (in the absence of friction):

$$x_1''(t) = -4\,x_1 + 4\,x_2$$
$$x_2''(t) = 2\,x_1 - 2\,x_2\,.$$

(8 points)

This is a second order system of DE's and it's a conservative system, so we search for a basis of solutions of the form $\cos(\omega\,t)\underline{\mathbf{v}}$, $\sin(\omega\,t)\underline{\mathbf{v}}$ *with* $\omega^2 = -\lambda$ *and* $A\,\underline{v} = \lambda\,\underline{\mathbf{v}}$, *in case* $\lambda < 0$, *and of the form* $\underline{\mathbf{v}}$, $t\underline{\mathbf{v}}$ *in case* $\lambda = 0$ *is an eigenvalue of* A *:*

$$|A - \lambda\,I| = \begin{vmatrix} -4-\lambda & 4 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda+4)(\lambda+2) - 8 = \lambda^2 + 6\,\lambda + 8 - 8 = \lambda(\lambda+6)\,.$$

$\lambda = 0$ *: Solve the homogenous system*

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right].$$

Since $1\ col_1 + 1\ col_2 = \underline{\mathbf{0}}$, $\underline{\mathbf{v}} = [1, 1]^T$ *is an eigenvector.*

$\lambda = -6$ $(\omega = \sqrt{6}\,)$ *: Solve the homogeneous system*

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right]$$

Since $2\ col_1 - 1\ col_2 = \underline{\mathbf{0}}$, $\underline{\mathbf{v}} = [2, -1]^T$ *is an eigenvector.*

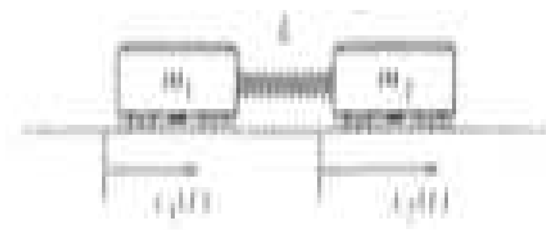
Thus the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 + c_2\,t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3\cos(\sqrt{6}\,t) + c_4\sin(\sqrt{6}\,t)\right)\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(Make sure you could describe the two fundamental "modes", and that you understand that by looking at the components of the vector expressions on the right you recover the motions $x_1(t), x_2(t)$ *of the two masses. Also that the four free parameters correspond to the solution space being 4−dimensional and the natural initial values that uniquely determine a solution: specifying initial displacement and velocity for each of* $x_1(t), x_2(t)$*.).*

2) If the Hooke's constant for the spring connecting the two cars is $k = 4000\,\frac{N}{m}$, then what are the masses of the two cars in order that their motion be governed by the system of differential equations above?

(2 points)



$$m_1 x_1''(t) = k(x_2 - x_1) = -4000\, x_1 + 4000\, x_2$$
$$m_2 x_2''(t) = -k(x_2 - x_1) = 4000\, x_1 - 4000\, x_2$$

In order to recover the system in (1), $\frac{4000}{m_1} = 4$ *and* $\frac{4000}{m_2} = 2$ *so* $m_1 = 1000\ kg,\ m_2 = 2000\ kg$. *Notice the fact that the second car is twice as massive as the first car is intuitively consistent with the fact that the oscillation amplitude of the first car is twice that of the second, in the problem 1 fundamental mode for which the masses are oscillating out of phase.*