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Student I.D.___________________

Math 2250−1
Quiz 11 Solutions
November 23, 2011

1a) Find the general solution $[x(t), y(t)]^T$ to the homogeneous system of differential equations

$$x'(t) = -4x + 2y$$
$$y'(t) = 4x - 2y.$$

(6 points)

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda+4) - 8 = \lambda^2 + 6\lambda + 8 - 8 = \lambda(\lambda+6)$$

So the eigenvalues are $\lambda = 0, \lambda = -6$.

Eigenvector for $\lambda = 0$:

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

Since $col_1 + 2\, col_2 = 0$, $\underline{\boldsymbol{v}} = [1, 2]^T$ is an eigenvector.

Eigenvector for $\lambda = -6$:

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$\underline{\boldsymbol{v}} = [1, -1]^T$.

Thus the general solution to the homogeneous first order linear systems of DE's is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

1b) Consider two tanks. Tank one contains 50 gallons of water, and tank two contains 100 gallons of water. Water flows from tank 1 to tank 2 through a one pipe, and from tank 2 back to tank 1 through another pipe. The flow rate in both pipes is 200 gallons per hour. There are no other inlet or outlet pipes. After some initial allocation of solute $x(0) = x_0$ to tank 1 and $y(0) = y_0$ to tank 2, let $x(t)$ and $y(t)$ denote the salt amounts for $t > 0$. Assume the water in each tank is well−mixed so that salt concentrations can be treated as uniform in each tank. Use this information and input−output analysis to derive the first order system of differential equations for $x(t)$ and $y(t)$. (Your answer is the system in part (a).)

(4 points)

$$x'(t) = r_i c_i - r_o c_o = 200\frac{y}{100} - 200\frac{x}{50} = -4x + 2y$$

$$y'(t) = r_i c_i - r_o c_o = 200\frac{x}{50} - 200\frac{y}{100} = 4x - 2y.$$

(and the initial conditions are $x(0) = x_0, y(0) = y_0$.)