Math 2250-1
Wed Nov. 2

- play with the RLC circuit, where the driving voltage is 120 hz. AC from a wall socket.

  the inductance is adjustable, and we'll exhibit "practical resonance"

  This is Tuesday's notes, page 4.
- Using total energy to find ω_0 in mechanical systems, page 5 Tuesday.

Then begin <u>Chapter 10: Laplace transform</u>

which ties in beautifully with chapter 5
(after 10, we'll cover 6, 7, 9.)

$|f(t)| \le Ce^{Mt}$

The Laplace transform is a linear transformation that transforms piecewise continuous functions $f(t)$, defined for $t \geq 0$ and with at most exponential growth into functions $\mathcal{L}\{f(t)\}(s) = F(s)$ defined and continuous for $s > M$.

$$\boxed{\mathcal{L}\{f(t)\}(s) := \int_0^{\infty} e^{-st} f(t)\,dt}$$

<u>Examples:</u>

<u>Exercise 1</u> : Use the definition of Laplace transform to show

a) $\mathcal{L}\{1\}(s) = \frac{1}{s}$ $\quad (s>0)$

b) $\mathcal{L}\{e^{\alpha t}\}(s) = \frac{1}{s-\alpha}$ $\quad (s>\alpha$, or if $\alpha = a+ik$, $s>a)$

c) the Laplace transformation is <u>linear</u>:

$$\mathcal{L}\{f_1(t)+f_2(t)\}(s) = F_1(s)+F_2(s)$$
$$\mathcal{L}\{cf(t)\}(s) = c\,F(s)$$

d) Use $\mathcal{L}\{e^{ikt}\}(s) = \frac{1}{s-ik}$ $\quad (s>0)$, from (b) + Euler, to show

$$\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$$
$$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$$

$\longrightarrow$ We're checking table entries from the front cover of our text

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t) \xrightarrow{\text{definition}}$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$	e^{at}	$\dfrac{1}{s-a}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		

Exercise 2 Use linearity and table (entries we've checked) to compute $\mathcal{L}\{7 - 5e^{2t} + 4\cos 2t\}(s)$

Exercise 3 It's a theorem (hard to prove) that a given Laplace transform $F(s)$ can arise from at most 1 piecewise continuous function $f(t)$ [well, the values of $f(t)$ at points of discontinuity don't affect $F(s)$]. So, you can use linearity and a good table to compute inverse Laplace transforms.

Find $\mathcal{L}^{-1}\left\{\frac{4}{s^2+4} - \frac{2s}{s^2+16}\right\}(t)$.

Laplace transforms and differential equations

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)dt}_{dv} = \underset{u}{e^{-st}} \underset{v}{f(t)}\Big]_0^\infty - \int_0^\infty \underset{v}{f(t)} \underset{du}{(-se^{-st})dt}$$

$du = -se^{-st}dt$
$v = f(t)$

$$= -f(0) + sF(s) \qquad (s > M).$$

so $\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$

so
$$\begin{aligned}\mathcal{L}\{f''(t)\}(s) &= s\,\mathcal{L}\{f'(t)\}(s) - f'(0)\\ &= s\left(sF(s) - f(0)\right) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

etc.

Connection to chapter 5:

Exercise 4 Let $x(t)$ be the solution to

$$\text{IVP} \begin{cases} x'' + 4x = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

non-resonant, forced, undamped oscillator.

Use Laplace transform to find $x(t)$ without $x = x_p + x_H$ & matching IC's to free parameters in x_H

Magic!! Laplace both sides of the DE equality:

$$s^2 X(s) - s\underbrace{x_0}_{=2} - \underbrace{v_0}_{=1} + 4(X(s)) = \frac{10s}{s^2+9}$$

$$X(s)(s^2+4) = \frac{10s}{s^2+9} + 2s+1$$

$$X(s) = \frac{10s}{(s^2+4)(s^2+9)} + \frac{2s+1}{s^2+4}$$

$$= 10s\left[\frac{1}{5}\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)\right] + \frac{2s+1}{s^2+4}$$

(Chapter 2 partial fractions trick)

$$= \frac{2s}{s^2+4} - \frac{2s}{s^2+9} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$X(s) = \frac{4s}{s^2+4} + \frac{1}{s^2+4} - \frac{2s}{s^2+9}$$

$\mathcal{L}^{-1}$:

$$x(t) = \underbrace{4\cos 2t + \frac{1}{2}\sin 2t}_{x_H} \underbrace{- 2\cos 3t}_{x_p}$$