

§5.6

Damped forced oscillations (c>0)

Math 2290-1
Tues Nov. 1

①

$$m x'' + c x' + k x = F_0 \cos \omega t$$

(1st finish Monday notes)

$$\begin{aligned} k (x_p &= A \cos \omega t + B \sin \omega t) \\ + c (x_p' &= -A \omega \sin \omega t + B \omega \cos \omega t) \\ + m (x_p'' &= -A \omega^2 \cos \omega t - B \omega^2 \sin \omega t) \end{aligned}$$

$$\begin{aligned} \mathcal{L}(x_p) &= \cos \omega t [kA + Bc\omega - mA\omega^2] + \sin \omega t [kB - Ac\omega - mB\omega^2] = \cos \omega t (F_0) \end{aligned}$$

$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\text{so } \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} k - m\omega^2 & -c\omega \\ c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} F_0(k - m\omega^2) \\ F_0 c\omega \end{bmatrix}$$

$$\text{so } A \cos \omega t + B \sin \omega t = C \cos(\omega t - \alpha)$$

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}$$

$\uparrow$ small if $\omega \approx \omega_0$ $\uparrow$ small if $c \propto 0$

$$x_H(t) = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & r_1, r_2 < 0 \text{ underdamped.} \\ e^{-\gamma t} (C_1 + C_2 t) & \text{critically damped} \\ e^{-\gamma t} (A \cos \omega_d t + B \sin \omega_d t) & \text{underdamped.} \end{cases}$$

So, full soln

$$x(t) = x_p(t) + x_H(t)$$

$\uparrow$
periodic

$\uparrow$
decays exponentially to zero regardless of choices of C_1, C_2 (or A, B).

So we call $x_p(t) = C \cos(\omega t - \alpha) = x_{sp}(t)$, steady periodic
 $x_H(t) = x_{tr}(t)$; transient sol'n, reflects initial values

example (worked out)

$$\begin{cases} x'' + 2x' + 26x = 82 \cos 4t \\ x(0) = 6 \\ x'(0) = 0 \end{cases}$$

could redo page 2 with numbers rather than letters (exam possibility),
but we can also plug in:

$$x_p(t) = A \cos 4t + B \sin 4t$$

$$\begin{aligned} m &= 1 \\ \omega &= 4 \\ c &= 2 \\ k &= 26 \end{aligned}$$

$$k - m\omega^2 = 26 - 16 = 10$$

$$c\omega = 8$$

$$F_0 = 82$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{100 + 64} \begin{bmatrix} 82 \cdot 10 \\ 82 \cdot 8 \end{bmatrix} = \begin{bmatrix} \frac{820}{164} \\ \frac{656}{164} \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$x_p(t) = 5 \cos 4t + 4 \sin 4t = \sqrt{41} \cos(4t - \alpha)$$



$$\alpha = \arctan(.8).$$

$x_H(t)$:

$$r^2 + 2r + 26 = 0$$

$$(r+1)^2 + 25 = 0$$

$$(r+1+5i)(r+1-5i) = 0$$

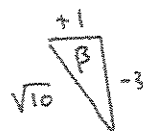
$$r = -1 \pm 5i$$

$$x_H(t) = e^{-t} (A \cos 5t + B \sin 5t)$$

$$x(t) = 5 \cos 4t + 4 \sin 4t + e^{-t} (A \cos 5t + B \sin 5t)$$

$$x(0) = 6 = 5 + A \Rightarrow A = 1$$

$$x'(0) = 0 = 0 + 16 + A + 5B = 15 + 5B \Rightarrow B = -3$$



$$x(t) = \sqrt{41} \cos(4t - \alpha) + e^{-t} \sqrt{10} \cos(5t - \beta)$$

$$\alpha = \arctan(.8)$$

$$\beta = \arctan(-3)$$

$$x_{sp}(t)$$

$$x_{tr}(t)$$

Practical resonance (see page 1)

$$x_{sp}(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

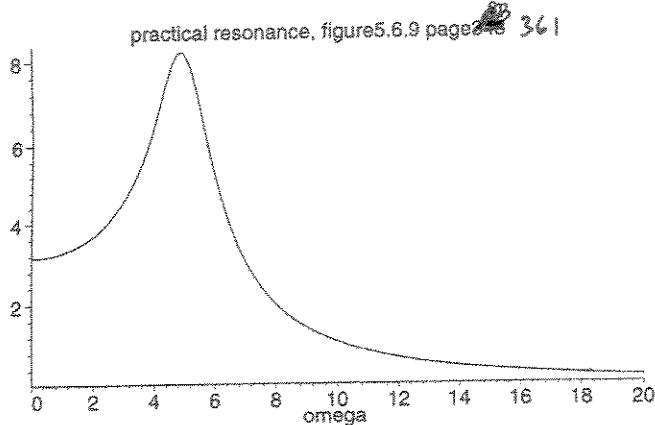
if c is small
and $\omega^2 \approx \omega_0^2 = \frac{k}{m}$ } denom tiny $\Rightarrow$ amplitude of x_{sp} huge (relative to F_0)
This is called practical resonance

Vary ω in page 3 example

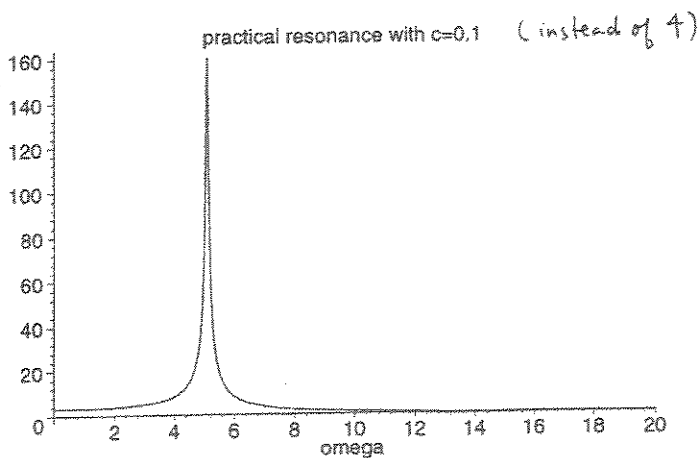
$$x'' + cx' + 26x = 82 \cos \omega t$$

$$C = \frac{82}{\sqrt{(26-\omega^2)^2 + c^2\omega^2}} = C(\omega)$$

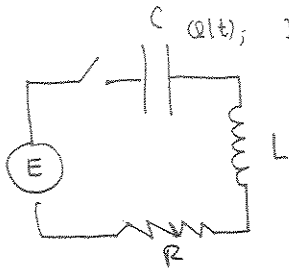
```
> plot(82/sqrt((26-omega^2)^2+4*omega^2), omega=0..20,
color=black, title='practical resonance, figure 5.6.9 page 361');
```



```
> plot(82/sqrt((26-omega^2)^2+.01*omega^2), omega=0..20,
color=black, title='practical resonance with c=0.1');
```



the mechanical ~ electrical analogy con't.



$$L I' + R I + \frac{Q}{C} = E = E_0 \sin \omega t$$

so

$$L I'' + R I' + \frac{1}{C} I = E_0 \omega \cos \omega t$$

$\tilde{\alpha}$ is the phase delay of current, relative to driving voltage

$$\tilde{\alpha} = \alpha \neq \pi/2$$

since $\sin(x + \pi/2) = \cos x$.

transcribe from page 1

$$I(t) = I_{sp}(t) + I_{tr}(t)$$



amplitude

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \tilde{\alpha})$$

$$\text{for } I_0 = \frac{E_0 \omega}{\sqrt{(\frac{1}{C} - L\omega^2)^2 + R^2 \omega^2}}$$

$$I_0 = \frac{\omega}{\omega} \frac{E_0}{\sqrt{(\frac{1}{\omega C} - L\omega)^2 + R^2}}$$

maximum response for fixed E_0, R ,
when

$$L\omega = \frac{1}{\omega C}$$

$$C = \frac{1}{L\omega^2}$$

$$L = \frac{1}{C\omega^2}$$

if C adjustable

if L adjustable.

old-style demo!

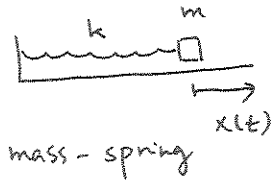
the denominator
 $Z = \sqrt{(\frac{1}{\omega C} - L\omega)^2 + R^2}$
is called the
impedance of
the circuit.

practical
resonance.

If you're an engineer concerned about resonance or practical resonance, you need to know how to figure out natural frequencies of mechanical systems.

The easiest way is often to use conservation of energy (essentially the antiderivative of Newton's law).

- We've done (via Newton)



Energy way:

$$KE + PE \equiv \text{const.}$$

$$\frac{1}{2} m (x'(t))^2 + \frac{1}{2} k x^2 \equiv \text{const}$$

↓

$$W = \int_0^x F(x) dx = \int_0^x k s ds = \left[\frac{1}{2} k s^2 \right]_0^x = \frac{1}{2} k x^2$$

take D_t :

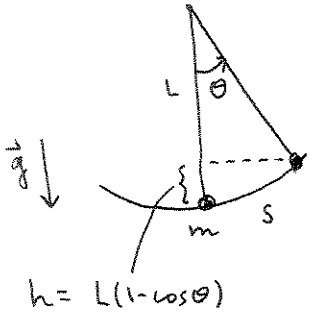
$$\frac{1}{2} m 2 x' x'' + \frac{1}{2} k 2 x x' \equiv 0$$

$$x' [m x'' + k x] \equiv 0$$

the DE!

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

- We've done (via KE+PE)



$$KE + PE \equiv \text{const}$$

$$\frac{1}{2} m s'(t)^2 + mg h \equiv \text{const}$$

$$\frac{1}{2} m (L \theta')^2 + mg L (1 - \cos \theta) \equiv \text{const}$$

D_t :

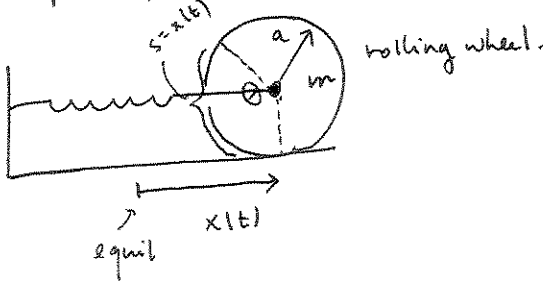
$$m L^2 \theta' \theta'' + mg L (\sin \theta) \theta' \equiv 0$$

$$L m \theta' [L \theta'' + g \sin \theta] \equiv 0$$

θ small $\Rightarrow L \theta'' + g \theta \equiv 0$

$$\theta'' + \frac{g}{L} \theta = 0 \quad \boxed{\omega_0 = \sqrt{\frac{g}{L}}}$$

- Example 4 page 353 (see also HW)



$$KE = \underbrace{\frac{1}{2} m (x')^2}_{\text{from motion of center of mass}} + \frac{1}{2} I \omega^2; \quad PE = \frac{1}{2} k x^2$$

$\omega = \frac{d\theta}{dt}$

moment of inertia = $\frac{m a^2}{2}$ for disk

rotational KE

$$x = a \theta$$

D_t :

$$x' = a \omega; \quad \omega = \frac{x'}{a}$$

$$KE + PE = \frac{1}{2} m (x')^2 + \frac{1}{2} \frac{1}{2} m a^2 \left(\frac{x'}{a} \right)^2 + \frac{1}{2} k x^2 = \frac{3}{4} m (x')^2 + \frac{1}{2} k x^2 \equiv \text{const}$$

$$\frac{1}{4} m (x')^2$$

D_t :

$$\frac{3}{2} m x' x'' + k x x' \equiv 0$$

$$x' \left[\frac{3}{2} m x'' + k x \right] \equiv 0$$

$$\boxed{\omega_0 = \sqrt{\frac{2k}{3m}}}$$