

Recall that problems which are underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

4.1–4.4 review: concepts related to linear combinations: independence, span, subspace, basis, dimension.

w8.1) Consider the matrix transformation, $T(\underline{x}) = A\underline{x}$ with matrix $A_{3 \times 4}$ given by

$$A := \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 4 & 9 & 0 \\ 2 & 2 & 8 & -4 \end{bmatrix}.$$

Notice that the input vectors \underline{x} are in \mathbf{R}^4 , i.e. the domain of the function T . The range of this function T is exactly the span of the columns, and is a subspace of \mathbf{R}^3 . The solution space to $A\underline{x} = 0$ is the subspace of the domain \mathbf{R}^4 consisting of the vectors which are transformed (squashed) to the zero vector in \mathbf{R}^3 . For this reason the homogeneous solution subspace is called the *nullspace* of T (or A).

Here is the reduced row echelon form of A :

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 7 & -8 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

w8.1a) Find a basis for the nullspace of T . What the dimension of the nullspace?

w8.1b) Find a basis for the range of T . Choose this basis by using appropriate columns of A . What is the dimension of the range?

w8.1c) The dimensions of the nullspace and range that you computed in parts (a),(b) should add up to 4. This is not an accident. Consider a general matrix transformation $T(\underline{x}) = B\underline{x}$ with domain \mathbf{R}^n and target space \mathbf{R}^m , i.e. the matrix is $B_{m \times n}$. Relate the dimension of the nullspace of T and the dimension of the range of T to the numbers of columns in the reduced row echeleon form of B with and without leading 1's. Use these dimension counts to explain why

$$\dim(\text{nullspace}(T)) + \dim(\text{range}(T)) = \dim(\text{domain}(T))$$

always holds. (This fact is known as the "rank+nullity" theorem, because the dimension of the nullspace is called the nullity, and the dimension of thre range is called the rank.)

w8.1d) Consider the special case of $T(\underline{x}) = B\underline{x}$ for $B_{3 \times 3}$, i.e. $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$. If the nullspace of T is a line through the origin, what geometric object must the range be? What are the other three possible geometric configurations for the nullspace and range, and how are they related to the number of leading 1's in the reduced row echelon form of B ?

5.1

solving initial value problems for linear homogeneous second order differential equations, given a basis for the solution space:

1, 9, **10, 14.**

superposition for linear differential equations, and its failure for non-linear DE's:

17, 27, 28 .

finding general solutions for constant coefficient homogeneous DE's by searching for exponential or related functions:

33, **34**, 39, **40 .**

5.2

testing collections of functions for dependence and independence:

1, **2, 4**, 7, **8**, 9, 11.

solving IVP's for homogeneous linear DE's given a basis of solutions:

13, **14**,

solving IVP's for non-homogeneous linear DE's given a particular solution and the general solution to the associated homogeneous DE

21, **22**,

superposition:

25, 26

Here's a problem that integrates the ideas from sections 5.1–5.2:

w8.2a) Show that the three functions

$$y_1(x) = \cos(2 \cdot x), \quad y_2(x) = \cos\left(2 \cdot x - \frac{\pi}{4}\right), \quad y_3(x) = \sin(2 \cdot x)$$

are all solutions to the differential equation

$$y'' + 4 \cdot y = 0 .$$

w8.2b) The differential equation above is a second order linear homogeneous DE, so the solution space is 2-dimensional. Thus the three functions y_1, y_2, y_3 above must be linearly dependent. Find a linear dependency. (Hint: use a trigonometry addition angle formula.)

w8.2c) Explain why $y_1(x), y_3(x)$ are a basis for the solution space to the DE in part (a).

w8.2d) Pick values for A, B so that $y(x) = A \cdot x + B$ is a particular solution to the non-homogeneous DE $y'' + 4 \cdot y = 8 \cdot x - 12$.

w8.2e) Solve the initial value problem

$$\begin{aligned} y'' + 4 \cdot y &= 8 \cdot x - 12 \\ y(0) &= 5 \\ y'(0) &= -2 \end{aligned}$$