Math 2250−1
Week 7 concepts and homework, due October 7.

Recall that problems which are underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

Because I assume you are proficient in computing reduced row echelon forms, determinants, and inverse matrices by hand, you are free to use technology for the computations in this week's homework, once you set up the problems. Remember that the basic matrix commands for Maple are illustrated in the "Maple commands" file posted on our homework page.

If you need practice, I'd recommend doing some of these computations by hand and only using technology to check your answers afterwards, however.

4.1: linear combinations of vectors in R^2 and R^3; linear dependendence and independence; subspaces of R^3 ;
1, 7, 9, 10, 15, 16, 22, 25, 26, 33.

w7.1) Consider the three vectors

$$\underline{\boldsymbol{u}} := \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \underline{\boldsymbol{v}} := \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \underline{\boldsymbol{w}} := \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

a) Compute the magnitudes (lengths) of $|\underline{\boldsymbol{u}}|$ and $|\underline{\boldsymbol{v}}|$.
b) Express $\underline{\boldsymbol{w}}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, by solving the appropriate linear system.
c) Make a careful and accurate sketch which illustrates your answer to (b), as we did in class on Friday September 30. (You can print off free graph paper at http://www.printfreegraphpaper.com/)
d) Find a linear combination of the three vectors which adds up to the zero vector. (You've already done the work for this in part (c)!) Illustrate this linear combination adding up to zero on your sketch for (c).

w7.2) Consider the three vectors

$$\underline{\boldsymbol{u}} := \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \underline{\boldsymbol{v}} := \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \quad \underline{\boldsymbol{w}} := \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} .$$

a) Compute the reduced row echelon form of the matrix $\langle \underline{\boldsymbol{u}} | \underline{\boldsymbol{v}} | \underline{\boldsymbol{w}} \rangle$ to deduce that every vector $\underline{\mathbf{b}}$ in R^3 can be expressed uniquely as a linear combination of these three vectors, i.e.

$$c_1 \underline{\boldsymbol{u}} + c_2 \underline{\boldsymbol{v}} + c_3 \underline{\boldsymbol{w}} = \underline{\boldsymbol{b}}$$

is always solvable for the linear combination coefficients c_1, c_2, c_3 (also called the coordinates of $\underline{\boldsymbol{b}}$ with respect to the basis $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}\}$) and these coordinates are unique. (In particular, the only linear combination of the vectors which adds up to $\underline{\boldsymbol{0}}$ is when $c_1 = c_2 = c_3 = 0$, so the vectors also satsify the definition of independence.)
b) Compute the determinant of the matrix you worked with in part (a), and explain why the result of (a) also follows from this computation.

w7.3) Consider the three vectors

$$\underline{\boldsymbol{u}} := \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \underline{\boldsymbol{v}} := \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \ \underline{\boldsymbol{w}} := \begin{bmatrix} 0 \\ 8 \\ -3 \end{bmatrix}.$$

a) Use a reduced row echelon computation to check that these vectors do not span R^3, and that in those rare cases that a vector $\underline{\boldsymbol{b}}$ is expressible as a linear combination of $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}$, the linear combination coefficients are never unique. (In particular, there are values of c_1, c_2, c_3 so that $c_1\underline{\boldsymbol{u}} + c_2\underline{\boldsymbol{v}} + c_3\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}}$, with not all of $c_1, c_2, c_3 = 0$, i.e. the vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}$ are linearly dependent.)
b) Verify your answer in (a) by computing the determinant of the matrix you worked with in part (a).
c) Use the reduced row echelon form of the matrix $[\underline{\boldsymbol{u}}|\underline{\boldsymbol{v}}|\underline{\boldsymbol{w}}]$ to express $\underline{\boldsymbol{w}}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$...as we did in class on September 30 and/or October 3.
d) Use your work from (c) to re−express the particular linear combination $\underline{\boldsymbol{u}} + 2\,\underline{\boldsymbol{v}} + 3\,\underline{\boldsymbol{w}}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ only.

4.2: *subspaces of R^n, expressed either as the span of a collection of vectors and/or as the solution space of a homogeneous matrix equation $Ax = 0$.*
3,**4**,5, **6**, **9**, 15, 18, 24, 27, 29;

w7.4) In problem **w7.3** you realized that $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}\} = span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$. Find the implicit equation of the plane of points $[x, y, z]$ these vectors span by computing the reduced row echelon form of the matrix below and interpreting the results. Explain your reasoning.

$$\begin{bmatrix} 1 & -3 & x \\ 2 & 2 & y \\ -1 & 0 & z \end{bmatrix}$$

w7.5) In problem **w7.4** you showed how to go from a spanning set for the plane to its implicit equation. In this problem, you'll do an example of the reverse procedure: Find two linearly independent vectors that span the plane with implicit equation

$$x + 2\cdot y + 3\cdot z = 0 .$$

Hint: The matrix for this very small homogeneous linear system is already in reduced row echelon form. ...backsolve!

w7.6) Consider the homogenous system

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & 2 & 8 \\ -1 & -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Express the solution space (which is a subspace of R^4) of this matrix equation as the span of two vectors.

4.3: *testing for independence of vectors; using reduced row echelon form to find dependencies for the columns of the matrix.*
1, 3, 6, 9, 10, 16, 17, 18, 23, 25.

w7.7) Consider the four columns of the matrix from problem **w7.6**. Since there are 4 of them and they are vectors in R^3 you know they must be dependent. But in this case, no three of them are even independent. Using the reduced row echelon form of this matrix, which you have already computed in **w7.6**, write down all possible choices of just 2 of the column vectors which will still span the same plane in R^3 that all four columns spanned. Hint: use the "magic" fact that column dependencies/independencies do not change when you do elementary row operations. (We discuss this fact in class, which only seems magic.)

4.4: *bases for vector spaces (including subspaces); creating a basis from a spanning set by deleting dependent vectors; adding vectors to an independent set to create a basis; dimension of a vector space.*
1, 2, 3, 6, 8, 9, 11, 13, 26.

w7.8)
a) Exhibit a basis for the plane spanned by the three vectors in **w7.3**. What is the dimension of this plane?
b) Find a third vector in R^3 to augment to the two vectors in your basis from part (a), to create a basis of three vectors for R^3. Justify your work.
c) Exhibit a basis for the solution space to the homogeneous matrix equation in **w7.6**. What is the dimension of this subspace of R^4?
d) Find two more vectors in R^4 to augment with the two vectors in your basis from part (c), to create a basis for R^4.
e) Exhibit a basis for the span of the four columns of the matrix in **w7.6**, made out of two of the original columns. What is the dimension of this subspace of R^3?

w7.9a) In class we'll discuss why doing the three "elementary vector" operations on a set of vectors, namely (i) interchanging their order, (ii) multiplying one of them by a nonzero constant, or (iii) replacing one of them by its sum with a multiple of another one, do not change the span of the set. Use this fact, and start with the original four column vectors of the matrix in **w7.6**, to find a basis for the plane they span which is of the particularly nice form

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ * \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ * \end{bmatrix} \right\}.$$

Hint: This amounts to computing the reduced column echelon form of the matrix in **w7.6**.
b) Express the two vectors you found in the **w7.8e** subspace basis as linear combinations of the two vectors you found in the different basis for the same subspace, in **w7.9a**. You should be able to read off the linear combination coefficients because of the nice form for the basis in **w7.9a**.