

Recall that problems which are underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

7.2-7.3 carry over problem from last week:

w14.1) Consider the non-homogeneous system

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Last week you solved the homogenous differential equation

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and found the general solution

$$\mathbf{x}_H(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This problem shows that the method of undetermined coefficients can work to find particular solutions for systems of differential equations, just as it did for single differential equations.

a) Use the method of undetermined coefficients to find a particular solution of the form

$$\mathbf{x}_p = t \cdot \mathbf{a} + \mathbf{b}$$

Hint: Plug $\mathbf{x}_p(t)$ into both sides of the non-homogeneous DE, which has the form $\mathbf{x}'(t) = A\mathbf{x} + t\mathbf{c} + \mathbf{d}$.

Since the functions $\{1, t\}$ are independent, their vector coefficients on each side of the differential equation must match. This will let you solve for \mathbf{a}, \mathbf{b} in terms of \mathbf{c}, \mathbf{d} (and A, A^{-1}).

b) An alternate approach to finding particular solutions is to use the Laplace transform: Transform each of the differential equations in the system and then solve for $X(s), Y(s)$ and use inverse Laplace transform, to find a particular solution with $x(0) = 0 = y(0)$. This particular solution will be consistent with your answer from part (a), but will likely have some terms corresponding to the homogeneous solution in order to satisfy the zero initial conditions. Feel free to use technology for the partial fractions decomposition of $X(s), Y(s)$, or work by hand if you prefer.

7.4) *Second order systems of differential equations arising from conservative systems. Identifying fundamental modes and natural angular frequencies; forced oscillation problems and the potential for practical resonance when the forcing frequency is close to a natural frequency.*

7.4: 2, 3, 8, 12, 13, 14, 16, 18.

w14.2) This is a continuation of 2. Now let's force the spring system in problem 2, with a sinusoidal force on the second mass at (variable) angular frequency ω , as in the slightly different text example on pages 440-442. Thus we consider the system

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F_0 \cos(\omega t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

a) Find a particular solution of the form

$$\mathbf{x}_p(t) = \cos(\omega t) \mathbf{c}.$$

Hint: Plug this guess into the differential equation. You will notice that each term simplifies to some vector times the function $\cos(\omega t)$. Thus, after you factor out the $\cos(\omega t)$ term you are left with a matrix equation to solve for $\underline{c} = \underline{c}(\omega)$. You will get formulas analogous to equations (34, 35) on page 441, except your c_1, c_2 will blow up at $\omega = 1, 3$, the natural frequencies for this problem.

b) The general solution to this forced oscillation problem is the particular solution from part (a), plus the general solution to the homogeneous problem, which you found in problem (2). In a physical problem with a slight amount of damping but the same masses and spring constants, the particular solution would be close to the one you found in part (a), and the homogeneous solutions would be close to the ones you found in problem 2, except that they would be (slowly) exponentially decaying because of the damping. Thus the particular solution would be the steady periodic solution, and the homogeneous solution would be transient. By plotting the magnitude $\|\underline{c}(\omega)\| = \sqrt{c_1(\omega)^2 + c_2(\omega)^2}$ as a function of ω , you create a "practical resonance" chart analogous to those we created in Chapter 5. Create such a plot, for the angular frequency range $0 \leq \omega \leq 5$. Use $F_0 = 1$. Your plot should look like Figure 7.4.10, except your magnitude function will peak at $\omega = 1, 3$.

w14.3) This is a continuation of **12**. As we discuss in class on Monday November 28, if a matrix is multiplied by a scalar c , then the eigenvectors of the new matrix are the same as the eigenvectors of the original matrix, but the corresponding eigenvalues are all multiplied by c . (This is the simple fact that if $A\underline{y} = \lambda\underline{y}$ then $(cA)\underline{y} = c\lambda\underline{y}$.)

a) Suppose the same configuration as in **12**, except that now we have each spring constant $k = 100m$. How are the new natural frequencies related to what they were in problem 12? How about the natural modes?

b) What if it was the reverse situation, i.e. heavy masses relative to the spring constants, so that $m = 100k$?

w14.4) This is a continuation of **18**. In physics you learn that you can recover the final velocities from the initial ones in a conservative problem like 18 by equating the initial momentum m_1v_0 to the final

momentum $m_1v_1 + m_2v_2$ and the initial kinetic energy $\frac{1}{2}m_1v_0^2$ to the final kinetic energy

$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$, and solving this system of equations for v_1 and v_2 . Carry this procedure out for the data in **18** and show that your answer agrees with your work in that problem.