Name....Solutions....................

I.D. number....................

Math 2250-1
FINAL EXAM
December 16, 2011

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	______	30
2	______	35
3	______	15
4	______	20
5	______	25
6	______	25
total	______	150

1) A motorboat containing the pilot has total mass of 200 kilograms, and its motor is able to provide 50 Newtons of thrust. However, when the boat is in motion drag from the water produces a force of 5 newtons for each meter/sec of boat velocity.

1a) Use Newton's law to explain why (while the motor is on) the boat velocity $v(t)$ satisfies the differential equation

$$v'(t) = \frac{1}{4} - \frac{v}{40}.$$

(5 points)

$m v' = \vec{F} = \underset{\text{force}}{\text{thrust}} + \underset{\text{force.}}{\text{drag}}$

$m = 200$

$200 v' = 50 - 5v$

$v' = \frac{50}{200} - \frac{5}{200} v$

$v' = \frac{1}{4} - \frac{1}{40} v$

1b) What is the equilibrium solution for the velocity v(t)? Is this solution stable or unstable? Explain with a phase diagram.

(5 points)

$\frac{1}{4} - \frac{v}{40} = 0$

$10 - v = 0$

$v = 10$ m/s is equil soltn for v.

$v' = \frac{1}{4} - \frac{v}{40} = \frac{1}{40}(10 - v)$

So $v = 10$ is (asymptotically) stable

1c) Solve the initial value problem for the boat's velocity, assuming the boat starts at rest, i.e.

$$v'(t) = \frac{1}{4} - \frac{v}{40}$$
$$v(0) = 0$$

As it turns out, we've learned four methods for solving a differential equation like this one. Use two of your favorites! If you try more than two ways indicate clearly which two are to be graded.

(20 points)

1st order linear

$v' + \frac{1}{40} v = \frac{1}{4}$

$e^{\frac{t}{40}}\left(v' + \frac{v}{40}\right) = \frac{1}{4} e^{t/40}$

$e^{t/40} v = \int \frac{1}{4} e^{t/40} dt = \frac{40}{4} e^{t/40} + c$

$e^{t/40} v = 10 e^{t/40} + C$

$v = 10 + Ce^{-t/40}$

$v(0) = 0 \Rightarrow C = -10$

$v = 10 - 10 e^{-t/40}$

separable

$v' = -\frac{1}{40}(v-10)$

$\frac{dv}{dt} = v' = -.025(v-10)$

$\frac{dv}{v-10} = -.025\, dt$

$\ln|v-10| = -.025t + \tilde{C}$

$v - 10 = Ce^{-.025t}$

$v = 10 + Ce^{-.025t}$

$v(0) = 0 \Rightarrow C = -10$

$v = 10 - 10e^{-.025t}$

Chapter 5 1st order linear.

$v' + .025 v = .25$

$v = v_p + v_H$. $\quad v_p = c \Rightarrow .025c = .25$

$\Rightarrow c = 10$

$v_p = 10.$

v_H: $p(r) = r + .025.$

$r = -.025$

$v_H(t) = c_1 e^{-.025t}$

$v = 10 + c_1 e^{-.025t}$

$v(0) = 0 \Rightarrow c_1 = -10$

$v = 10 - 10 e^{-.025t}$

Laplace

$v' + .025v = .25$

$\mathcal{L}$: $sV(s) - \cancel{v_0}^{0} + .025V(s) = \frac{.25}{s}$

$V(s)[s + .025] = \cancel{v_0}^{0} + \frac{.25}{s}$

$V(s) = \frac{.25}{s(s+.025)}$

$= .25\left[\frac{1}{s} - \frac{1}{s+.025}\right]\frac{1}{.025}$

$= 10\left[\frac{1}{s} - \frac{1}{s+.025}\right]$

$\mathcal{L}^{-1}$: $v(t) = 10 - 10e^{-.025t}$

2) Consider the following initial value problem, which could arise from Newton's second law in a forced mass-spring oscillation problem:

$$x''(t) + 4x'(t) + 13x(t) = 26$$
$$x(0) = 0$$
$$x'(0) = 4\,.$$

2a) If the applied force in this problem is 52 N, then what are the Hooke's constant, mass, and damping coefficient? Include correct units. (5 points)

$$mx'' + cx' + kx = F = 52 \quad N$$

$$\Rightarrow \quad \frac{m}{2}x'' + \frac{c}{2}x' + \frac{k}{2}x = 26$$

comparing, deduce $m = 2$ kg.
$c = 8$ kg/s
$k = 26$ N/m

2b) Solve this initial problem using the methods of Chapter 5, based on particular and homogeneous solutions. (15 points)

x_H: $e^{rt} \to p(r) = r^2 + 4r + 13 = 0$
$= (r+2)^2 + 9 = 0$
$(r+2)^2 = -9$
$r+2 = \pm 3i$
$r = -2 \pm 3i$

$\Rightarrow x_H(t) = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t.$

x_p: try $x_p = c \Rightarrow 13c = 26 \Rightarrow c = 2.$

So $x = x_p + x_H$
$x(t) = 2 + c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$

$x(0) = 0 = 2 + c_1 \qquad \Rightarrow c_1 = -2$
$x'(0) = 4 = -2c_1 + 3c_2 \qquad \Rightarrow 4 = 4 + 3c_2 \Rightarrow c_2 = 0.$

$$\boxed{\text{so} \quad x(t) = 2 - 2e^{-2t}\cos 3t}$$

(long computation of $x'(t)$:
$x'(t) = c_1\left[-2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t\right] + c_2\left[-2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t\right]$
so $x'(0) = -2c_1 + 3c_2$

2c) Use Laplace transform techniques to re-solve the same initial value problem

$$x''(t) + 4x'(t) + 13x(t) = 26$$
$$x(0) = 0$$
$$x'(0) = 4.$$

(15 points)

$$s^2 X(s) - s x_0 - v_0 + 4(sX(s) - x_0) + 13X(s) = \frac{26}{s}.$$

$$s^2 X(s) - 4 + 4sX(s) + 13X(s) = \frac{26}{s}$$

$$X(s)[s^2+4s+13] = \frac{26}{s} + 4$$

$$X(s) = \frac{26}{s(s^2+4s+13)} + \frac{4}{s^2+4s+13}$$

$$= \frac{A}{s} + \frac{Bs+c}{s^2+4s+13}$$

$$26 = A(s^2+4s+13) + Bs^2 + Cs$$

@ $s=0$: $26 = 13A \Rightarrow A=2.$

$$26 = 2(s^2+4s+13) + Bs^2 + Cs$$

coeffs of s: $0 = 8 + C \Rightarrow C = -8$

s^2: $0 = B+2 \Rightarrow B = -2$

$$X(s) = \frac{2}{s} + \frac{-2s-8}{s^2+4s+13} + \frac{4}{s^2+4s+13}$$

$$X(s) = \frac{2}{s} + \frac{-2s-4}{s^2+4s+13}$$

$$= \frac{2}{s} + \frac{-2(s+2)}{(s+2)^2+9}$$

$$\boxed{x(t) = 2 - 2e^{-2t}\cos 3t}$$

or

usually doesn't pay to recombine before parfrac, but some of you may have done this:

$$X(s) = \frac{26+4s}{s(s^2+4s+13)}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$$

$$\Rightarrow 26+4s = A(s^2+4s+13) + s(Bs+C)$$

$$\Rightarrow 26+4s = s^2[A+B] + s[4A+c] + 1(13A).$$

1: $26 = 13A \Rightarrow A = 2$

s: $4 = 4A + C = 8 + C \Rightarrow C = -4$

s^2: $0 = A + B \Rightarrow B = -2$

$$\Rightarrow X(s) = \frac{2}{s} + \frac{-2s-4}{s^2+4s+13}$$

as at left.

3) Consider the matrix

$$A := \begin{bmatrix} -1 & 0 & 2 \\ 1 & -6 & 0 \\ 0 & 6 & -2 \end{bmatrix}$$

3a) Find the characteristic polynomial and factor it to find the eigenvalues of A. (Hint, the eigenvalues you get should be $= 0, -4, -5$; your job is to derive these values.) (5 points)

$$\begin{vmatrix} -1-\lambda & 0 & 2 \\ 1 & -6-\lambda & 0 \\ 0 & 6 & -2-\lambda \end{vmatrix}$$

det expand across 1st row

$$= (-1-\lambda)\begin{vmatrix} -6-\lambda & 0 \\ 6 & -2-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & -6-\lambda \\ 0 & 6 \end{vmatrix}$$

$$= -(\lambda+1)(\lambda+6)(\lambda+2) + 12$$

$$= -(\lambda+1)[\lambda^2+8\lambda+12] + 12$$

$$= -\lambda^3 + \lambda^2[-1-8] + \lambda[-8-12] \cancel{-12} \cancel{+12}$$

$$= -\lambda^3 - 9\lambda^2 - 20\lambda = -\lambda[\lambda^2+9\lambda+20]$$

$$= -\lambda(\lambda+5)(\lambda+4)$$

so roots are $\lambda = 0, -5, -4$.

3b) An eigenvector for $\lambda = 0$ is given by $\underline{u} = [6, 1, 3]^T$, and an eigenvector for $\lambda = -4$ is given by $\underline{v} = [-2, -1, 3]^T$. Find an eigenvector for $\lambda = -5$. (10 points)

$\lambda = -5$:

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 6 & 3 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \frac{1}{2}R_1 \\ \frac{1}{3}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$col_1 + col_2 - 2\,col_3 = 0$

$$\boxed{\vec{v} = [1, 1, -2]^T}$$

or,

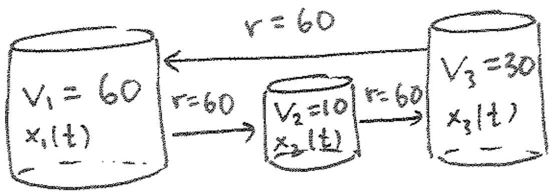
$$\begin{array}{c} \\ \frac{1}{2}R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = -\frac{1}{2}t$
$v_2 = -\frac{1}{2}t$
$v_3 = t$

$$\rightarrow \vec{v} = t\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

choose $\vec{v} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, for example.

4) Consider the following three-tank configuration. In tank one there is uniformly mixed volume of 60 gallons pf water, and $x_1(t)$pounds of salt solute. In tank two there is mixed volume of 10 gallons and $x_2(t)$ pounds of salt. In tank three there is 30 gallons of liquid and $x_3(t)$ pounds of salt. Water is pumped slowly from tank one to tank two, from tank two to tank three, and from tank three back to tank one, and all rates are 60 gallons per hour.



4a) Model this input-output model, to arrive at the first order system of differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -6 & 0 \\ 0 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. = \begin{bmatrix} -x_1 + 2x_3 \\ x_1 - 6x_2 \\ 6x_2 - 2x_3 \end{bmatrix}$$

(5 points)

$$x_1' = r_i c_i - r_o c_o = 60\frac{x_3}{30} - 60\frac{x_1}{60} = -x_1 + 2x_3$$

$$x_2' = r_i c_i - r_o c_o = 60\frac{x_1}{60} - 60\frac{x_2}{10} = x_1 - 6x_2$$

$$x_3' = r_i c_i - r_o c_o = 60\frac{x_2}{10} - 60\frac{x_3}{30} = 6x_2 - 2x_3$$

4b) Use your work from problem (3), in which you found an $\boldsymbol{R}^3$ basis of eigenvectors for the matrix in the 4a system of differential equations, to find the general solution.

(10 points)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$\lambda = 0$ $\vec{u} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$ $\quad$ $\lambda = -4$ $\vec{v} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$ $\quad$ $\lambda = -5$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$,

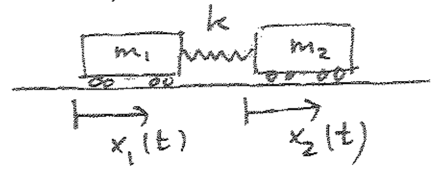
4c) Based on your solution to 4b, and assuming there is a positive amount of salt in the system at time $t = 0$, what happens to the relative salt amounts in each tank as t approaches infinity? Why does your answer make sense even without referring to the solution in 4b?

(5 points)

$x_1 : x_2 : x_3 \to 6 : 1 : 3$ because as $t \to \infty$ $e^{-4t}, e^{-5t} \to 0$.

this makes sense because the concentration should become constant throughout the system (mass/volume), and the volume ratios are $60 : 10 : 30$, i.e. $6:1:3$.

5a) Consider a "train" of two cars of masses m_1, m_2 connected by spring with Hooke's constant k as indicated in the diagram below. Assuming no friction and no external forces, derive the system of two differential equations for the displacements $x_1(t), x_2(t)$ of the two cars from equilibrium.



(4 points)

$$m_1 x_1'' = k(x_2 - x_1)$$
$$m_2 x_2'' = -k(x_2 - x_1)$$

$$\downarrow$$

$$x_1'' = -\frac{k}{m_1} x_1 + \frac{k}{m_1} x_2$$
$$x_2'' = \frac{k}{m_2} x_1 - \frac{k}{m_2} x_2$$

5b) If the Hooke's constant for the spring connecting the two cars is $k = 3000 \, \frac{N}{m}$, then determine the masses of the two cars so that the differential equations above reduce to

$$x_1''(t) = -3x_1 + 3x_2$$
$$x_2''(t) = x_1 - x_2$$

(4 points)

$$\frac{k}{m_1} = 3$$
$$\Rightarrow m_1 = \frac{k}{3} = 1000 \text{ kg.}$$

$$\frac{k}{m_2} = 1$$
$$\Rightarrow m_2 = k = 3000 \text{ kg}$$

5c) Find the general solution to the system in (5b), repeated below for your convenience:

(10 points)

$$x_1''(t) = -3x_1 + 3x_2$$
$$x_2''(t) = x_1 - x_2$$

$$|A-\lambda I| = \begin{vmatrix} -3-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+3)(\lambda+1) - 3 = \lambda^2 + 4\lambda + 3 - 3$$
$$= \lambda(\lambda+4).$$

roots $\lambda = 0, -4$.

$\lambda = 0$: $\left.\begin{matrix} -3 & 3 \\ 1 & -1 \end{matrix}\right| \begin{matrix} 0 \\ 0 \end{matrix}$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

if $A\vec{v} = \vec{0}$ then $(c_1 + c_2 t)\vec{v}$ satisfies $\vec{x}'' = \vec{0} = A\vec{x}$.

$\lambda = -4$: $\left.\begin{matrix} 1 & 3 \\ 1 & 3 \end{matrix}\right| \begin{matrix} 0 \\ 0 \end{matrix}$

$\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

$\omega = 2$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3 \cos 2t + c_4 \sin 2t)\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

5d) Solve the initial value problem for the system of differential equations above, with

$$x_1(0) = 0$$
$$x_1'(0) = 8$$
$$x_2(0) = 0$$
$$x_2'(0) = 0$$

and describe the resulting motion of the two cars as a superposition of motions related to two "fundamental modes" (in this problem one of the modes isn't actually vibrating).

(7 points)

$$x_1(t) = c_1 + c_2 t - 3c_3 \cos 2t - 3c_4 \sin 2t$$
$$x_1'(t) = c_2 + 6c_3 \sin 2t - 6c_4 \cos 2t$$
$$x_2(t) = c_1 + c_2 t + c_3 \cos 2t + c_4 \sin 2t$$
$$x_2'(t) = c_2 - 2c_3 \sin 2t + 2c_4 \cos 2t$$

$x_1(0) = 0 = c_1 - 3c_3$
$x_1'(0) = 8 = c_2 - 6c_4$
$x_2(0) = 0 = c_1 + c_3$
$x_2'(0) = 0 = c_2 + 2c_4.$

$\Rightarrow c_1 = c_3 = 0$

$c_2 - 6c_4 = 8$
$c_2 + 2c_4 = 0$

$E_1 - E_2 \Rightarrow -8c_4 = 8$

so $c_4 = -1$; $c_2 - 2 = 0$
$c_2 = 2.$

velocity of each car $= 2$

out of phase osc

1st car amplitude $= 3$, 2nd car $= 1$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin 2t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

soln is superposition of cars moving ~~tog~~ at constant velocity of 2 (1st term), with an out of phase oscillation of angular freq. 2, in with 1st car has 3 times amplitude of 2nd car.

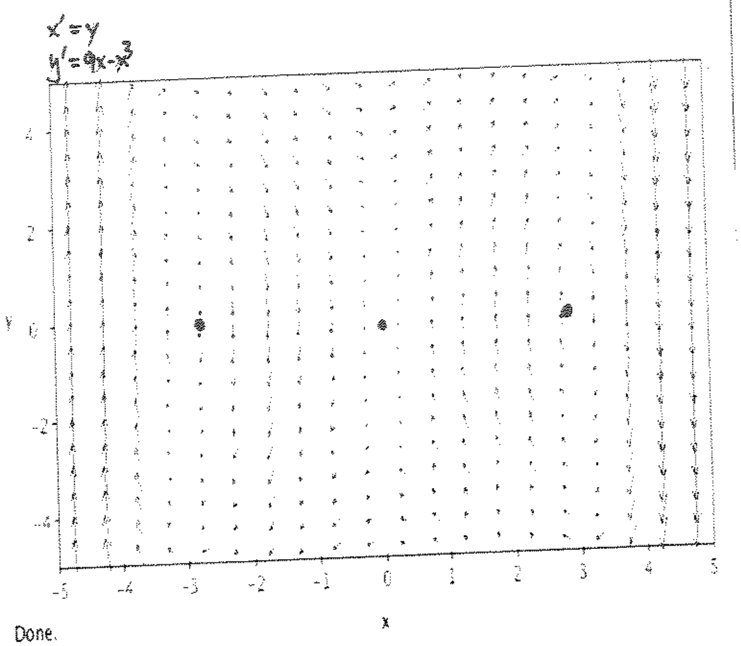
6) We studied various non-linear mechanical oscillation problems in Chapter 9. For example, we studied some non-linear mass-spring models. Here is an interesting one, in which the spring force has the opposite sign from the usual one as long as the displacements $x(t)$ from equilibrium are small:

$$x''(t) = 9x - x^3 .$$

Converting this as usual to a first order system yields the system

$$x'(t) = y$$
$$y'(t) = 9x - x^3 .$$

Here's a phase portrait for this system:



easy to find, but not asked:
$y = 0$
$9x - x^3 = 0$
$x(9 - x^2) = 0.$

6a) It's relatively straightforward to find the three equilibrium solutions. They are $(0, 0)$, $(-3, 0)$, $(3, 0)$. Use the Jacobian matrix to classify these equilbria, as much as is possible. Explain.

(10 points)

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ 9-3x^2 & 0 \end{bmatrix}$$

@ $x=0$, $y=0$ $\quad J = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 9 & -\lambda \end{vmatrix}$$
$$= \lambda^2 - 9$$
$$= (\lambda - 3)(\lambda + 3)$$

(unstable) saddle

$$J(\pm 3, 0) = \begin{bmatrix} 0 & 1 \\ -18 & 0 \end{bmatrix}$$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -18 & -\lambda \end{vmatrix} = \lambda^2 + 18 = 0$$
$$\lambda = \pm i\, 3\sqrt{2}$$

stable center for linearized problem but indeterminate for non-linear

6b) Find the solution $[u(t), v(t)]^T$ to the linearized system at the equilibrium point $(0, 0)$. Use this solution (and the relevant eigenvalues and eigenvectors) to sketch a qualitatively accurate picture of what the phase portrait to both the linearized and the original non-linear system of differential equations looks like near the origin. (Of course, your sketch should be consistent with the pplane picture on the previous page, but should include more relevant details.)

(10 points)

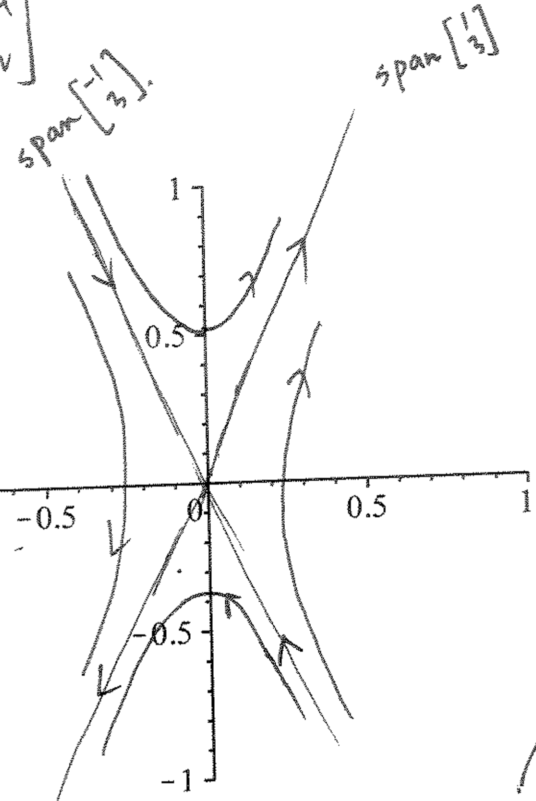
$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$\lambda = 3$ $\qquad$ $\lambda = -3$

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 9 & -3 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 9 & 3 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



So
$TE = KE + PE$
$= \frac{1}{2}(x')^2 - \frac{9}{2}x^2 + \frac{x^4}{4}.$

$x'' = F = 9x - x^3$

alternate to below, via physics, if $m=1$
$KE = \frac{1}{2}(x')^2.$
$PE = W = -\int_0^x 9x - x^3\,dx$
$= -\frac{9}{2}x^2 + \frac{x^4}{4}.$

6c) This problem is an example of one in which one can use a conserved "total energy" function to understand stability questions and solution trajectories. Use separation of variables or some other method to find a function of x, y which is constant for every solution $[x(t), y(t)]^T$ to the original system. What are the implications of this computation for the stability of the critical points $(\pm 3, 0)$?

(5 points)

$x' = y$
$y' = 9x - x^3$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{9x - x^3}{y}$$

$y\,dy = (9x - x^3)\,dx$

$\frac{1}{2}y^2 = \frac{9}{2}x^2 - \frac{x^4}{4} + C.$

$\frac{1}{2}y^2 - \frac{9}{2}x^2 + \frac{x^4}{4} = C$

for solns $x(t), y(t)$.

if you can show that this energy function has global minimum value at $(\pm 3, 0)$, then the equilibria must be stable, since solutions starting nearby will have energy close to this global min. value and could not leave this neighborhood while keeping their total energy constant

[one could use calculus or look at a graph of $E(x,y) = \frac{1}{2}y^2 - \frac{9}{2}x^2 + \frac{x^4}{4}$ to verify that the minima occur for $y=0$ (clear), and $x = \pm 3$].

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$	e^{at}	$\dfrac{1}{s-a}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		