Name Solutions

Student I.D.

Math 2250–1
Exam 2
November 10, 2011

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right–hand margin. **Good Luck!**

1) Here is a matrix and its reduced row echelon form:

$$A := \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 5 & 5 & 0 \\ -1 & 1 & 2 & -1 & 3 & 0 \\ 2 & 0 & -2 & 4 & -2 & 0 \\ -2 & 2 & 4 & -2 & 5 & 0 \end{array}\right] \quad \text{reduced row echelon form of A:} \quad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \atop \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1a) Find the general solution to the homogeneous matrix equation $A\underline{x} = \underline{0}$. (10 points)

$x_1 = s - 2t$
$x_2 = -s - t$
$x_3 = s$
$x_4 = t$
$x_5 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

1b) What is the dimension of the solution space to $A\underline{x} = \underline{0}$? Explain. (5 points)

2. The vectors $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are a basis

(they span the solution space. they are also linearly independent

1c) Consider the non–homogeneous matrix equation $A\underline{x} = \underline{b}$, where $\underline{b}$ is the fifth column of the matrix A. Use this information to find a particular solution for $A\underline{x} = \underline{b}$. Then use your previous work to write down the general solution to $A\underline{x} = \underline{b}$. (5 points)

$\vec{x}_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, Since $A\vec{x}_p = \begin{bmatrix} 5 \\ 3 \\ -2 \\ 5 \end{bmatrix}$.

Then the general soltn to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \vec{x}_p + \vec{x}_H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

2) Consider the linear homogeneous differential equation for $y(x)$:

$$y''(x) - 2\,y'(x) + 5\,y(x) = 0.$$

2a) Find the general solution to this differential equation, in the form $y(x) = c_1 \cdot y_1(x) + c_2 \cdot y_2(x)$.

(8 points)

$y = e^{rx} \longrightarrow p(r) = r^2 - 2r + 5 = 0$

$(r-1)^2 + 4 = 0$

$(r-1)^2 = -4$

$r - 1 = \pm 2i$

$\boxed{r = 1 \pm 2i}$

or quad. form:

$r = \dfrac{2 \pm \sqrt{4-20}}{2}$

$= 1 \pm \dfrac{4i}{2} = 1 \pm 2i$

So $\boxed{y(x) = c_1 e^{+x} \cos 2x + c_2 e^{x} \sin 2x}$

2b) Use your general solution from part (a) to show why the initial value problem

$$y''(x) - 2\,y'(x) + 5\,y(x) = 0.$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

always has a solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$. (In other words, find the linear system which you need to solve to find c_1, c_2 for given b_1, b_2 and verify that this system has a unique solution.)

(8 points)

$y(0) = b_0 = c_1$

$y'(0) = b_1 = c_1 + 2c_2$

$\left(y'(x) = c_1 e^{x} \cos 2x + c_2 e^{x} \sin 2x + c_1 e^{x}(-2\sin 2x) + c_2 (e^{x}(+2\cos 2x) \right)$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\uparrow$

Wronskian matrix at $x=0$, has inverse since $\det W = 2$

in fact, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$.

2c) What is the dimension of the solution space to the homogeneous differential equation in this problem? How is your answer (which you may know from general facts) related to your specific work in part (b)?

(4 points)

dimension $= 2$, since $y_1(x) = e^{x}\cos 2x$, $y_2(x) = e^{x}\sin 2x$ are a basis: since we can solve every IVP with $c_1y_1 + c_2y_2$ they ~~are a basis~~ span sol'n space

since if $c_1y_1 + c_2y_2 \equiv 0$ function

$b_0 = b_1$ for zero fcn, so by eqtn in 2b), $c_1 = c_2 = 0$ also

so these y_1, y_2 are independent

$\nearrow$

in general, dim sol'n space for n^{th} order linear homog. DE. is n.

2d) Use superposition and the method of undetermined coefficients to find a particular solution for the non–homogeneous differential equation

$$L(y) := y''(x) - 2\,y'(x) + 5\,y(x) = 20 + 10\,e^{2x}.$$

(10 points)

for $L(y) = 20$, try a const, see that $y_p = 4$ works.

for $L(y) = 10e^{2x}$ try

$\times 5\,[\, y_p = Ae^{2x} \,]$

$-2\,[\, y_p' = 2Ae^{2x} \,]$

$+1\,[\, y_p'' = 4Ae^{2x} \,]$

$L(y_p) = Ae^{2x}[5 - 4 + 4]$

$= 5Ae^{2x}$

So choose $A = 2$.

$\boxed{y_p = 2e^{2x}}$

so $L\left(4 + 2e^{2x}\right) = 20 + 10e^{2x}$

3) Consider the unforced damped oscillator equation

$$x''(t) + 4x'(t) + 3x(t) = 0 .$$

3a) Find the general solution to this homogeneous differential equation.

(10 points)

$p(r) = r^2 + 4r + 3 = (r+3)(r+1)$; roots $r = -3, -1$

$x_H(t) = c_1 e^{-3t} + c_2 e^{-t}$

3b) What sort of damping is exhibited by your solutions to part (a)?

(2 points)

overdamped. (two distinct negative real roots to $p(r)$; soltn passes through equilibrium at most once.)

3c) Now consider the forced damped harmonic oscillator equation with the same left hand side operator:

$$x''(t) + 4x'(t) + 3x(t) = -30\sin(3t)$$

A particular solution to this non–homongeneous differential equation is given by

$$x_p(t) = 2\cos(3t) + \sin(3t) .$$

(You do not need to check this fact; you can just use it.) Use this particular solution and your work from part (a) to solve the initial problem for this forced differential equation, with initial conditions

$$x(0) = 2$$
$$x'(0) = -1$$

(6 points)

$x(t) = x_p(t) + x_H(t)$

$x(t) = 2\cos 3t + \sin 3t + c_1 e^{-3t} + c_2 e^{-t}$

$x'(t) = -6\sin 3t + 3\cos 3t - 3c_1 e^{-3t} - c_2 e^{-t}$

$x(0) = 2 = 2 + c_1 + c_2$

$x'(0) = -1 = 3 - 3c_1 - c_2$

$c_1 + c_2 = 0$

$3c_1 + c_2 = 4$

$c_2 = -c_1 \Rightarrow 3c_1 - c_1 = 4$

$2c_1 = 4$

$\boxed{c_1 = 2, c_2 = -2}$

$\boxed{x(t) = 2\cos 3t + \sin 3t + 2e^{-3t} - 2e^{-t}}$

$x(0) = 2$ ✓

$x'(0) = -1$ ✓

3d) Identify the steady periodic and transient parts of your solution in part (c).

(2 points)

$x_{sp}(t) = 2\cos 3t + \sin 3t$

$x_{tr}(t) = 2e^{-3t} - 2e^{-t}$

4a) Use Laplace transforms (and the table you've been provided) to solve the forced oscillator equation

$$x''(t) + \omega_0^2 \cdot x(t) = \frac{F_0}{m} \cdot \cos(\omega_0 \cdot t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

(16 points)

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m} \frac{s}{s^2+\omega_0^2}$$

$$X(s)\left[s^2+\omega_0^2\right] = \frac{F_0}{m} \frac{s}{s^2+\omega_0^2} + s x_0 + v_0$$

$$X(s) = \frac{F_0}{m} \frac{s}{(s^2+\omega_0^2)^2} + x_0 \frac{s}{s^2+\omega_0^2} + \frac{v_0}{\omega_0} \frac{\omega_0}{s^2+\omega_0^2}$$

Table:

$$\boxed{x(t) = \frac{F_0}{m} \frac{t}{2\omega_0} \sin \omega_0 t + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t}$$

$k = \omega_0$

4b) What word describes the sort of behavior exhibited by solutions to this differential equation? (4 points)

resonance

5) Find the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+6s+25}+\frac{5}{s^2+3s+2}\right\}(t).$$

(10 points)

$$\frac{2s}{s^2+6s+25} = \frac{2s}{(s+3)^2+16} = \frac{2(s+3)-6}{(s+3)^2+16}$$

$$= \frac{2(s+3)}{(s+3)^2+16} + \frac{-6}{4}\,\frac{4}{(s+3)^2+16}$$

↑ in table
$a=-3$
$k=4$

So $\mathcal{L}^{-1}\left\{\frac{2s}{s^2+6s+25}\right\}(t) = 2e^{-3t}\cos 4t - \frac{3}{2}e^{-3t}\sin 4t$

$$\frac{5}{s^2+3s+2} = \frac{5}{(s+2)(s+1)} = 5\left(\frac{1}{s+1}-\frac{1}{s+2}\right)$$

So $\mathcal{L}^{-1}\left\{\frac{5}{s^2+3s+2}\right\}(t) = 5\left(e^{-t}-e^{-2t}\right).$

(you could also complete the square and use cosh, sinh, and translation entries).

thus

$$\boxed{\mathcal{L}^{-1}\left\{\frac{2s}{s^2+6s+25}+\frac{5}{s^2+3s+2}\right\}(t) = 2e^{-3t}\cos 4t - \frac{3}{2}e^{-3t}\sin 4t + 5e^{-t} - 5e^{-2t}}$$

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$	e^{at}	$\dfrac{1}{s-a}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		