Name_____

Student I.D.

Math 2250-1 Exam #1 September 29, 2011 SOLUTIONS

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions. There are 100 points possible. The point values for each problem are indicated in the right-hand margin. Good Luck!

Score

POSSIBLE

1	30
2	_25
3	25
4	20

TOTAL_____ 100

1) Suppose that an object moves vertically, subject only to the acceleration of gravity $g = 32 \frac{ft}{s^2}$ and a drag force proportional to the object's velocity. Choose the positive *y* direction to be up. For particular values of the object's mass and the drag coefficient, the differential equation

$$\frac{dv}{dt} = -32 - 0.2 \cdot v$$

governs the object's velocity v(t).

1a) Use Newton's second law and compare it to the differential equation above to deduce the mass *m* of the object if the drag coefficient is 1 *lb* per $\frac{ft}{s}$ of velocity. How is the object's mass related to its weight? (5 points)

Newton says mass times acceleration is net forces ... for a linear drag problem this yields

$$m \cdot \frac{dv}{dt} = -mg - k \cdot v$$

We are told that k = 1 lb per $\frac{ft}{s}$. So

$$m \cdot \frac{dv}{dt} = -mg - v$$
$$\frac{dv}{dt} = -g - \frac{1}{m} \cdot v$$

Comparing with the original differential equation we see that $\frac{1}{m} = 0.2$, so m = 5. The units of mass in the English system are slugs, and the weight w in pounds equals mg = 32 m.

1b) Use a phase diagram to determine $\lim_{t \to \infty} v(t)$ for all solutions to this differential equation

$$\frac{dv}{dt} = -32 - 0.2 \cdot v \,.$$

(10 points)

The equilibrium solution (which is the terminal velocity in linear drag problems) is the constant solution, i.e. the root of

$$-32 - 0.2 \cdot v = 0 \implies -32 = .2 \cdot v \implies v = -\frac{32}{.2} = -32 \cdot 5 = -160$$

If v > -160 then $-32 - .2 \cdot v < 0$ so v is decreasing and if v < 160 then $-32 - .2 \cdot v > 0$ so v is increasing Thus the phase diagram is

$$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow -160 \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$$
.

Therefore all solutions converge to $v = -160 \frac{ft}{s}$, the terminal velocity..

1c) Solve the following initial value problem for the differential equation above,

$$\frac{dv}{dt} = -32 - 0.2 \cdot v$$
$$v(0) = 40$$

Note that the object is initially thrown upwards.

This is a linear DE (also separable, but easier to use linear algorithm),

$$\frac{dv}{dt} + 0.2 \cdot v = -32$$

$$e^{0.2 \cdot t} \left(\frac{dv}{dt} + 0.2 \cdot v \right) = -32 \cdot e^{0.2 \cdot t},$$

$$\frac{d}{dt} \left(e^{0.2 \cdot t} v \right) = -32 \cdot e^{0.2 \cdot t},$$

$$e^{0.2 \cdot t} \cdot v = \int -32 \cdot e^{0.2 \cdot t} dt = -160 \cdot e^{0.2 \cdot t} + C .$$

$$v(0) = 40 \Rightarrow C = 200$$

$$divide \ by \ e^{0.2 \cdot t} \Rightarrow v(t) = -160 + 200 \cdot e^{-0.2 \cdot t} .$$

1d) When is the height y(t) of the object at its maximum, if its velocity satisfies the initial value problem in (1c)?

$$When v(t) = 0, i.e.$$

$$-160 + 200 \cdot e^{-0.2 \cdot t} = 0 \implies 200 \cdot e^{-0.2 \cdot t} = 160 \implies e^{-.02 \cdot t} = \frac{4}{5} = 0.8 \implies -0.2 \cdot t = \ln(0.8) \implies t = -5$$

$$\cdot \ln(0.8)$$

seconds.

(The decimal value is about 1.1 seconds .)

(10 points)

2) Consider the following input-output model: A large tank initially contains 10 gallons of very salty water - there are 10 pounds of salt dissolved in those 10 gallons. At time t = 0 less salty water begins to flow into the tank, at a rate of 10 gallons per minute. This water contains 0.1 pounds of salt per gallon. Also at time t = 0 water begins to flow out of the tank, at a rate of 5 gallons per minute.

2a) What is the volume of water in the tank at time t, at least before the tank fills up?

(5 points)

Water is flowing in at 10 gallons per minute, and flowing out at 5 gallons per minute, so the volume V(t) is increasing at 5 gallons per minute, $\frac{dV}{dt} = 5$. Since V(0)=10, deduce (integrate) that $V(t) = 10 + 5 \cdot t$.

2b) Assume the water is well-mixed, so that the salt concentration in the tank is uniform. Use your modeling ability to show that the pounds x(t) of salt in the tank at time t satisfies the differential equation (10 points)

$$\frac{dx}{dt} = 1 - \frac{5}{10 + 5 \cdot t} \cdot x(t)$$
$$\frac{dx}{dt} = r_i \cdot c_i - r_o \cdot c_o.$$

The concentration in the tank (and the concentration of the water flowing out) is
$$\frac{x(t)}{V(t)} \frac{lb}{gal}$$
. Thus

$$\frac{dx}{dt} = 10 \cdot (0.1) - 5 \cdot \frac{x(t)}{10 + 5 \cdot t} = 1 - \frac{5}{10 + 5 \cdot t} \cdot x(t) ,$$

2c) Solve the initial value problem for this differential equation (2b), i.e.

$$\frac{dx}{dt} = 1 - \frac{5}{10 + 5 \cdot t} \cdot x(t)$$

$$x(0) = 10$$
(10 points)

.

$$\frac{dx}{dt} + \frac{5}{10 + 5 \cdot t} \cdot x(t) = 1.$$
The integrating factor is the exponential of and antiderivative

$$\int \frac{5}{10+5\cdot t} dt = \ln(10+5\cdot t) \; .$$

But $e^{\ln(10 + 5 \cdot t)} = 10 + 5 \cdot t$, so

$$(10+5\cdot t) \cdot \left(\frac{dx}{dt} + \frac{5}{10+5\cdot t} \cdot x(t)\right) = (10+5\cdot t)\cdot 1,$$
$$\frac{d}{dt} \left((10+5\cdot t) \cdot x(t)\right) = 10+5\cdot t$$
$$(10+5\cdot t) \cdot x(t) = 10\cdot t + \frac{5}{2}\cdot t^{2} + C.$$

At t = 0, x = 10, so $10 \cdot 10 = C$. *Thus*

$$x(t) = \frac{10 \cdot t + \frac{5}{2} \cdot t^2 + 100}{10 + 5 \cdot t}$$

3) Consider the differential equation

$$\frac{dx}{dt} = 2x^2 - 6x$$

3a) Find the equilibrium solutions for this differential equation and use a phase portrait to determine their stability.

(10 points)

$$\frac{dx}{dt} = 2 \cdot \left(x^2 - 3 \cdot x\right) = 2 \cdot x \cdot \left(x - 3\right)$$

Thus the equilibrium solutions are x = 0 and x = 3. Evaluating $\frac{dx}{dt}$ between the equilibria gives ++++0---3++++++

so the phase portrait is

$$\rightarrow \rightarrow \rightarrow \rightarrow 0 \leftarrow \leftarrow \leftarrow \leftarrow 3 \rightarrow \rightarrow \rightarrow \rightarrow$$

So x = 0 is (asymptotically) stable and x = 3 is unstable.

3b) If this differential equation was modeling a population x(t), would it be a logistic or doomsday/extinction model? Briefly justify your answer.

(2 points)

Doomsday-extinction: the positive equilibrium solution is unstable. If initial values are between 0 and 3 then population dies out (extinction). If initial population is greater than 3 then population will converge to infinity in finte time (doomsday).

3c) Solve the initial value problem for this differential equation, with x(0) = 3, i.e. solve

$$\frac{dx}{dt} = 2x^2 - 6x.$$
$$x(0) = 4.$$

Hint: The partial fraction identity

$$\frac{1}{(x-a)\cdot(x-b)} = \frac{1}{b-a}\cdot\left(\frac{1}{x-b} - \frac{1}{x-a}\right)$$

might be useful.

(10 points)

$$\frac{dx}{x \cdot (x-3)} = 2 \cdot dt$$
$$-\frac{1}{3} \cdot \left(\frac{1}{x} - \frac{1}{x-3}\right) = 2 \cdot dt$$
$$\left(\frac{1}{x} - \frac{1}{x-3}\right) = -6 \cdot dt$$

integrate:

$$\ln(|x|) - \ln(|x-3|) = -6 \cdot t + C$$
$$\ln\left(\left|\frac{x}{x-3}\right|\right) = -6 \cdot t + C$$

exponentiate:

$$\left|\frac{x}{x-3}\right| = e^{C} \cdot e^{-6 \cdot t}$$
$$\frac{x}{x-3} = C \cdot e^{-6 \cdot t} \quad \text{(different } C\text{)}$$

use initial values:

$$4 = C$$

$$\frac{x}{x-3} = 4 \cdot e^{-6 \cdot t}$$

$$x = 4 \cdot e^{-6 \cdot t} (x-3) = 4 \cdot e^{-6 \cdot t} \cdot x - 12 \cdot e^{-6 \cdot t}$$

$$x(1-4 \cdot e^{-6 \cdot t}) = -12 \cdot e^{-6 \cdot t}$$

$$x = -\frac{12 \cdot e^{-6 \cdot t}}{1-4 \cdot e^{-6 \cdot t}} = \frac{12 \cdot e^{-6 \cdot t}}{4 \cdot e^{-6 \cdot t} - 1} = \frac{12}{4 - e^{6 \cdot t}}$$

3d) If you look at your solution to (3c) carelessly it may appear that x(t) exists for all positive t, and that there is a limiting value for x(t) as $t \to \infty$. What really happens, and when does it happen?

(3 points)

$$x(t) = \frac{12}{4 - e^{6 \cdot t}}$$

and as t increases from 0, the denominator decreases from 3, down to zero when $4 - e^{6 \cdot t} = 0$. Thus at this time x(t) will have increased to $+\infty$. We can solve the time of doomsday, namely

$$e^{6 \cdot t} = 4 \implies t = \frac{\ln(4)}{6}$$
.

4) Consider the matrix

$$A := \left[\begin{array}{rrrr} 1 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & -1 & 1 \end{array} \right]$$

4a) Compute the determinant of A.

(5 points)

e.g. expand across the first row, to get $|A| = 1 \cdot 4 + 2 \cdot (-2) + 0 = 0$.

4b) What does your computation in 4a tell you about the reduced row echelon form of *A*? About whether A^{-1} exists? About whether the matrix equation Ax = b has more than one solution *x*, no solutions *x*, or unique solutions *x*? (Here *x* and *b* are both column vectors.)

(5 points)

Since |A| = 0, the reduced row echelon form of A is not the identity matrix I. Thus A^{-1} does not exist. Thus the matrix equation Ax=b does not have unique solutions - it either has no solutions or infinitely many solutions.

4c) Find all solutions to

1	-2	0]	x		0	
-2	3	1		У	=	0	
0	-1	1		z		0	

(10 points)

We compute the reduced row echelon form of the augmented matrix. Actually, because the right hand side vector is zero, the fourth column of the augmented matrix will be zero at all stages and we don't really need to include it in our computation.

		1 -	2 0	0	
	-2	2	3 1	0	
	(- 0	1 1	0	
$R2 + 2 \cdot R1 \text{ in } R2 \rightarrow$	L				
	[1	-2	0	0	
	0	- 1	1	0	
	0	- 1	1	0	
$-R2$ in R2, $-R2 + R3$ in R3 \rightarrow	L		I		
	[1	-2	0	0	
	0	1	- 1	0	
	0	0	0	0	

 $R1 + 2 \cdot R2$ in $R1 \rightarrow$

1	0	-2	0
0	1	-1	0
0	0	0	0

 $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ Thus we may take z as a free parameter, and backsolve for x and y: $z = t, y = t, x = 2 \cdot t$, t any real number.

In vector form this reads

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

and is a parametric (explicit) description of a line with velocity vector $[2, 1, 1]^T$.