Name ____

Student I.D. ___________________________________

Math 2250-1
Exam #1
September 29, 2011

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

Score

	POSSIBLE
1________________________	30
2_________________________	25
3________________________	25
4________________________	20
TOTAL______________________	100

1) Suppose that an object moves vertically, subject only to the acceleration of gravity $g = 32\ \frac{ft}{s^2}$ and a drag force proportional to the object's velocity. Choose the positive y direction to be up. For particular values of the object's mass and the drag coefficient, the differential equation

$$\frac{dv}{dt} = -32 - 0.2 \cdot v$$

governs the object's velocity $v(t)$.

1a) Use Newton's second law and compare it to the differential equation above to deduce the mass m of the object if the drag coefficient is 1 lb per $\frac{ft}{s}$ of velocity. How is the object's mass related to its weight?

(5 points)

1b) Use a phase diagram to determine $\lim_{t \to \infty} v(t)$ for all solutions to this differential equation

$$\frac{dv}{dt} = -32 - 0.2 \cdot v \, .$$

(10 points)

1c) Solve the following initial value problem for the differential equation above,

$$\frac{dv}{dt} = -32 - 0.2 \cdot v$$

$$v(0) = 40\ .$$

(10 points)

Note that the object is initially thrown upwards.

1d) When is the height $y(t)$ of the object at its maximum, if its velocity satisfies the initial value problem in (1c)?

(5 points)

2) Consider the following input-output model: A large tank initially contains 10 gallons of very salty water - there are 10 pounds of salt dissolved in those 10 gallons. At time $t = 0$ less salty water begins to flow into the tank, at a rate of 10 gallons per minute. This water contains 0.1 pounds of salt per gallon. Also at time $t = 0$ water begins to flow out of the tank, at a rate of 5 gallons per minute.

2a) What is the volume of water in the tank at time t, at least before the tank fills up?

(5 points)

2b) Assume the water is well-mixed, so that the salt concentration in the tank is uniform. Use your modeling ability to show that the pounds $x(t)$ of salt in the tank at time t satisfies the differential equation

(10 points)

$$\frac{dx}{dt} = 1 - \frac{5}{10 + 5 \cdot t} \cdot x(t)$$

2c) Solve the initial value problem for this differential equation (2b), i.e.

$$\frac{dx}{dt} = 1 - \frac{5}{10 + 5 \cdot t} \cdot x(t)$$

$$x(0) = 10$$

(10 points)

3) Consider the differential equation

$$\frac{dx}{dt} = 2x^2 - 6x.$$

3a) Find the equilibrium solutions for this differential equation and use a phase portrait to determine their stability.

(10 points)

3b) If this differential equation was modeling a population $x(t)$, would it be a logistic or doomsday/extinction model? Briefly justify your answer.

(2 points)

3c) Solve the initial value problem for this differential equation, with $x(0) = 4$, i.e. solve

$$\frac{dx}{dt} = 2\,x^2 - 6\,x.$$

$$x(0) = 4\,.$$

Hint: The partial fraction identity

$$\frac{1}{(x-a)\cdot(x-b)} = \frac{1}{b-a}\cdot\left(\frac{1}{x-b} - \frac{1}{x-a}\right)$$

might be useful.

(10 points)

3d) If you look at your solution to (3c) carelessly it may appear that $x(t)$ exists for all positive t, and that there is a limiting value for $x(t)$ as $t \to \infty$. What really happens, and when does it happen?

(3 points)

4) Consider the matrix

$$A := \begin{bmatrix} 1 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

4a) Compute the determinant of A.

(5 points)

4b) What does your computation in 4a tell you about the reduced row echelon form of A? About whether A^{-1} exists? About whether the matrix equation $Ax = b$ has more than one solution x, no solutions x, or unique solutions x? (Here x and b are both column vectors.)

(5 points)

4c) Find all solutions to

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(10 points)