7.5 Application Engineering Functions

Periodic piecewise linear functions occur so frequently as input functions in engineering applications that they are sometimes called **engineering functions.** Computations with such functions are readily handled by computer algebra systems. In *Mathematica*, for instance, we can define

```
SawTooth[t_] := 2 t - 2 Floor[t] - 1
TriangularWave[t_] := Abs[2 SawTooth[(2 t - 1)/4]] - 1
SquareWave[t_] := Sign[ TriangularWave[t] ]
```

Plot each of the functions to verify that it has period 2 and that its name is aptly chosen. For instance, the result of

```
Plot[ SquareWave[t], {t, 0, 6}];
```

should look like Fig. 7.5.9. If $f(t)$ is one of these engineering functions and $p > 0$, then the function $f(2t/p)$ will have period p. To illustrate this, try

```
Plot[ TriangularWave[ 2 t/p ], {t, 0, 3 p}];
```

with various values of p.

Now let's consider the mass-spring-dashpot equation

```
diffEq = m x''[t] + c x'[t] + k x[t] == input
```

with selected parameter values and an input forcing function with period p and amplitude F_0.

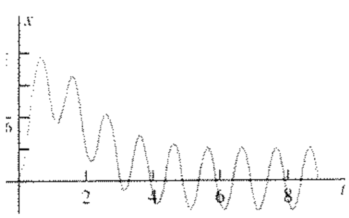
```
m = 4; c = 8; k = 5; p = 1; F0 = 4;
input = F0 SquareWave[2 t/p];
```

You can plot this **input** function to verify that it has period 1:

```
Plot[ input, {t, 0, 10}];
```

Finally, let's suppose that the mass is initially at rest in its equilibrium position and solve numerically the resulting initial value problem.

```
response = NDSolve[ {diffEq, x[0] == 0, x'[0] == 0},
               x, {t, 0, 10}, MaxSteps -> 1000 ]
Plot[ x[t] /. response, {t, 0, 10}];
```



'IGURE 7.5.18. Response $x(t)$) period 1 square wave input.

In the resulting Fig. 7.5.18 we see that after an initial transient dies out, the response function $x(t)$ settles down (as expected?) to a periodic oscillation with the same period as the input.

Investigate this initial value problem with several mass–spring–dashpot parameters—for instance, selected digits of your student ID number—and with input engineering functions having various amplitudes and periods.

7.6 Impulses and Delta Functions

Consider a force $f(t)$ that acts only during a very short time interval $a \leqq t \leqq b$, with $f(t) = 0$ outside this interval. A typical example would be the *impulsive force* of a bat striking a ball—the impact is almost instantaneous. A quick surge of voltage (resulting from a lightning bolt, for instance) is an analogous electrical phenomenon. In such a situation it often happens that the principal effect of the force depends only on the value of the integral

$$p = \int_a^b f(t)\,dt \tag{1}$$

and does not depend otherwise on precisely how $f(t)$ varies with time t. The number p in Eq. (1) is called the **impulse** of the force $f(t)$ over the interval $[a, b]$.

In the case of a force $f(t)$ that acts on a particle of mass m in linear motion, integration of Newton's law

$$f(t) = mv'(t) = \frac{d}{dt}[mv(t)]$$

yields

$$p = \int_a^b \frac{d}{dt}[mv(t)]\,dt = mv(b) - mv(a). \tag{2}$$

Thus the impulse of the force is equal to the change in momentum of the particle. So if change in momentum is the only effect with which we are concerned, we need know only the impulse of the force; we need know neither the precise function $f(t)$ nor even the precise time interval during which it acts. This is fortunate, because in a situation such as that of a batted ball, we are unlikely to have such detailed information about the impulsive force that acts on the ball.

Our strategy for handling such a situation is to set up a reasonable mathematical model in which the unknown force $f(t)$ is replaced with a simple and explicit force that has the same impulse. Suppose for simplicity that $f(t)$ has impulse 1 and acts during some brief time interval beginning at time $t = a \geqq 0$. Then we can select a fixed number $\epsilon > 0$ that approximates the length of this time interval and replace $f(t)$ with the specific function

$$d_{a,\epsilon}(t) = \begin{cases} \dfrac{1}{\epsilon} & \text{if } a \leqq t < a + \epsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

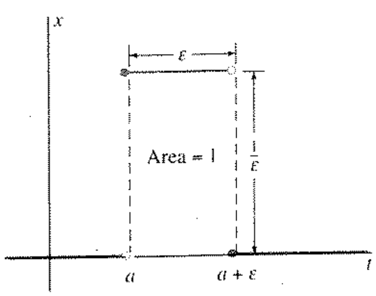


FIGURE 7.6.1. The graph of the impulse function $d_{a,\epsilon}(t)$.

This is a function of t, with a and ϵ being parameters that specify the time interval $[a, a+\epsilon]$. If $b \geqq a + \epsilon$, then we see (Fig. 7.6.1) that the impulse of $d_{a,\epsilon}$ over $[a, b]$ is

$$p = \int_a^b d_{a,\epsilon}(t)\,dt = \int_a^{a+\epsilon} \frac{1}{\epsilon}\,dt = 1.$$

Thus $d_{a,\epsilon}$ has a *unit* impulse, whatever the number ϵ may be. Essentially the same computation gives

$$\int_0^\infty d_{a,\epsilon}(t)\,dt = 1. \tag{4}$$

Because the precise time interval during which the force acts seems unimportant, it is tempting to think of an *instantaneous impulse* that occurs precisely at the instant $t = a$. We might try to model such an instantaneous unit impulse by taking the limit as $\epsilon \to 0$, thereby defining

$$\delta_a(t) = \lim_{\epsilon \to 0} d_{a,\epsilon}(t), \tag{5}$$

where $a \geqq 0$. If we could also take the limit under the integral sign in Eq. (4), then it would follow that

$$\int_0^\infty \delta_a(t)\,dt = 1. \tag{6}$$

But the limit in Eq. (5) gives

$$\delta_a(t) = \begin{cases} +\infty & \text{if } t = a, \\ 0 & \text{if } t \neq a. \end{cases} \tag{7}$$

Obviously, no actual function can satisfy both (6) and (7)—if a function is zero except at a single point, then its integral is not 1 but zero. Nevertheless, the symbol $\delta_a(t)$ is very useful. However interpreted, it is called the **Dirac delta function** at a after the British theoretical physicist P. A. M. Dirac (1902–1984), who in the early 1930s introduced a "function" allegedly enjoying the properties in Eqs. (6) and (7).

The following computation motivates the meaning that we will attach here to the symbol $\delta_a(t)$. If $g(t)$ is continuous function, then the mean value theorem for integrals implies that

$$\int_a^{a+\epsilon} g(t)\,dt = \epsilon g\left(\bar{t}\right)$$

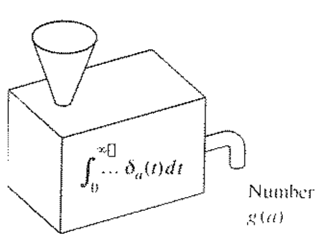
for some point $\bar{t}$ in $[a, a+\epsilon]$. It follows that

$$\lim_{\epsilon\to 0}\int_0^\infty g(t)d_{a,\epsilon}(t)\,dt = \lim_{\epsilon\to 0}\int_a^{a+\epsilon} g(t)\cdot\frac{1}{\epsilon}\,dt = \lim_{\epsilon\to 0} g\left(\bar{t}\right) = g(a) \tag{8}$$

by continuity of g at $t = a$. If $\delta_a(t)$ *were* a function in the strict sense of the definition, and if we could interchange the limit and the integral in Eq. (8), we therefore could conclude that

$$\int_0^\infty g(t)\delta_a(t)\,dt = g(a). \tag{9}$$

iction



GURE 7.6.2. A diagram strating how the delta function ts out" the value $g(a)$.

We take Eq. (9) as the *definition* (!) of the symbol $\delta_a(t)$. Although we call it the delta function, it is not a genuine function; instead, it specifies the *operation*

$$\int_0^\infty \cdots \delta_a(t)\,dt$$

which—when applied to a continuous function $g(t)$—sifts out or selects the value $g(a)$ of this function at the point $a \geqq 0$. This idea is shown schematically in Fig. 7.6.2. Note that we will use the symbol $\delta_a(t)$ only in the context of integrals such as that in Eq. (9), or when it will appear subsequently in such an integral.

For instance, if we take $g(t) = e^{-st}$ in Eq. (9), the result is

$$\int_0^\infty e^{-st}\delta_a(t)\,dt = e^{-as}. \tag{10}$$

We therefore *define* the Laplace transform of the delta function to be

$$\mathcal{L}\{\delta_a(t)\} = e^{-as} \quad (a \geqq 0). \tag{11}$$

If we write

$$\delta(t) = \delta_0(t) \quad \text{and} \quad \delta(t-a) = \delta_a(t), \tag{12}$$

then (11) with $a = 0$ gives

$$\mathcal{L}\{\delta(t)\} = 1. \tag{13}$$

Note that if $\delta(t)$ were an actual function satisfying the usual conditions for existence of its Laplace transform, then Eq. (13) would contradict the corollary to Theorem 2 of Section 7.1. But there is no problem here; $\delta(t)$ is not a function, and Eq. (13) is our *definition* of $\mathcal{L}\{\delta(t)\}$.

Delta Function Inputs

Now, finally, suppose that we are given a mechanical system whose response $x(t)$ to the external force $f(t)$ is determined by the differential equation

$$Ax'' + Bx' + Cx = f(t). \tag{14}$$

To investigate the response of this system to a unit impulse at the instant $t = a$, it seems reasonable to replace $f(t)$ with $\delta_a(t)$ and begin with the equation

$$Ax'' + Bx' + Cx = \delta_a(t). \tag{15}$$

But what is meant by the solution of such an equation? We will call $x(t)$ a solution of Eq. (15) provided that

$$x(t) = \lim_{\epsilon \to 0} x_\epsilon(t), \tag{16}$$

where $x_\epsilon(t)$ is a solution of

$$Ax'' + Bx' + Cx = d_{a,\epsilon}(t). \tag{17}$$

Because

$$d_{a,\epsilon}(t) = \frac{1}{\epsilon}\left[u_a(t) - u_{a+\epsilon}(t)\right] \tag{18}$$

is an ordinary function, Eq. (17) makes sense. For simplicity suppose the initial conditions to be $x(0) = x'(0) = 0$. When we transform Eq. (17), writing $X_\epsilon = \mathcal{L}\{x_\epsilon\}$, we get the equation

$$(As^2 + Bs + C)X_\epsilon(s) = \frac{1}{\epsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s}\right) = \left(e^{-as}\right)\frac{1 - e^{-s\epsilon}}{s\epsilon}.$$

If we take the limit in the last equation as $\epsilon \to 0$, and note that

$$\lim_{\epsilon \to 0} \frac{1 - e^{-s\epsilon}}{s\epsilon} = 1$$

by l'Hôpital's rule, we get the equation

$$(As^2 + Bs + C)X(s) = e^{-as}, \tag{19}$$

if

$$X(s) = \lim_{\epsilon \to 0} X_\epsilon(x).$$

Note that this is precisely the same result that we would obtain if we transformed Eq. (15) directly, using the fact that $\mathcal{L}\{\delta_a(t)\} = e^{-as}$.

On this basis it is reasonable to solve a differential equation involving a delta function by employing the Laplace transform method exactly as if $\delta_a(t)$ were an ordinary function. It is important to verify that the solution so obtained agrees with the one defined in Eq. (16), but this depends on a highly technical analysis of the limiting procedures involved; we consider it beyond the scope of the present discussion. The formal method is valid in all the examples of this section and will produce correct results in the subsequent problem set.

Example 1 A mass $m = 1$ is attached to a spring with constant $k = 4$; there is no dashpot. The mass is released from rest with $x(0) = 3$. At the instant $t = 2\pi$ the mass is struck with a hammer, providing an impulse $p = 8$. Determine the motion of the mass.

Solution According to Problem 15, we need to solve the initial value problem

$$x'' + 4x = 8\delta_{2\pi}(t); \quad x(0) = 3, \quad x'(0) = 0.$$

We apply the Laplace transform to get

$$s^2X(s) - 3s + 4X(s) = 8e^{-2\pi s},$$

so

$$X(s) = \frac{3s}{s^2+4} + \frac{8e^{-2\pi s}}{s^2+4}.$$

Recalling the transforms of sine and cosine, as well as the theorem on translations on the t-axis (Theorem 1 of Section 7.5), we see that the inverse transform is

$$\begin{aligned} x(t) &= 3\cos 2t + 4u(t-2\pi)\sin 2(t-2\pi) \\ &= 3\cos 2t + 4u_{2\pi}(t)\sin 2t. \end{aligned}$$

Because $3\cos 2t + 4\sin 2t = 5\cos(2t - \alpha)$ with $\alpha = \tan^{-1}(4/3) \approx 0.9273$, separation of the cases $t < 2\pi$ and $t \geqq 2\pi$ gives

$$x(t) \approx \begin{cases} 3\cos 2t & \text{if } t \leqq 2\pi, \\ 5\cos(2t - 0.9273) & \text{if } t \geqq 2\pi. \end{cases}$$

The resulting motion is shown in Fig. 7.6.3. Note that the impulse at $t = 2\pi$ results in a visible discontinuity in the velocity at $t = 2\pi$, as it instantaneously increases the amplitude of the oscillations of the mass from 3 to 5. ■

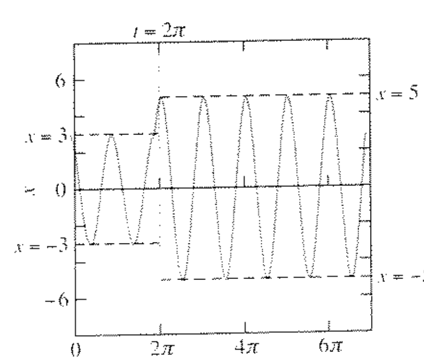


FIGURE 7.6.3. The motion of the mass of Example 1.

Delta Functions and Step Functions

It is useful to regard the delta function $\delta_a(t)$ as the derivative of the unit step function $u_a(t)$. To see why this is reasonable, consider the continuous approximation $u_{a,\epsilon}(t)$ to $u_a(t)$ shown in Fig. 7.6.4. We readily verify that

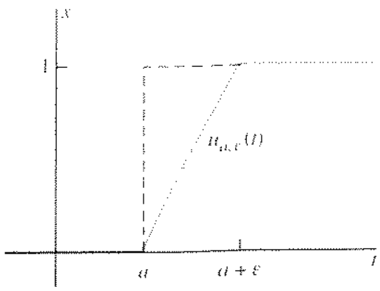


FIGURE 7.6.4. Approximation of $u_a(t)$ by $u_{a,\epsilon}(t)$.

$$\frac{d}{dt}u_{a,\epsilon}(t) = d_{a,\epsilon}(t).$$

Because

$$u_a(t) = \lim_{\epsilon \to 0} u_{a,\epsilon}(t) \quad \text{and} \quad \delta_a(t) = \lim_{\epsilon \to 0} d_{a,\epsilon}(t),$$

an interchange of limits and derivatives yields

$$\frac{d}{dt}u_a(t) = \lim_{\epsilon \to 0} \frac{d}{dt}u_{a,\epsilon}(t) = \lim_{\epsilon \to 0} d_{a,\epsilon}(t),$$

and therefore

$$\frac{d}{dt}u_a(t) = \delta_a(t) = \delta(t-a). \tag{20}$$

We may regard this as the *formal definition* of the derivative of the unit step function, although $u_a(t)$ is not differentiable in the ordinary sense at $t = a$.

Example 2 We return to the *RLC* circuit of Example 5 of Section 7.5, with $R = 110\ \Omega$, $L = 1$ H, $C = 0.001$ F, and a battery supplying $e_0 = 90$ V. Suppose that the circuit is initially passive—no current and no charge. At time $t = 0$ the switch is closed and at time $t = 1$ it is opened and left open. Find the resulting current $i(t)$ in the circuit.

Solution In Section 7.5 we circumvented the discontinuity in the voltage by employing the integrodifferential form of the circuit equation. Now that delta functions are available, we may begin with the ordinary circuit equation

$$Li'' + Ri' + \frac{1}{C}i = e'(t).$$

In this example we have

$$e(t) = 90 - 90u(t-1) = 90 - 90u_1(t),$$

so $e'(t) = -90\delta(t-1)$ by Eq. (20). Hence we want to solve the initial value problem

$$i'' + 110i' + 1000i = -90\delta(t-1); \quad i(0) = 0, \quad i'(0) = 90. \tag{21}$$

The fact that $i'(0) = 90$ comes from substitution of $t = 0$ in the equation

$$Li'(t) + Ri(t) + \frac{1}{C}q(t) = e(t)$$

with the numerical values $i(0) = q(0) = 0$ and $e(0) = 90$.

When we transform the problem in (21), we get the equation

$$s^2I(s) - 90 + 110sI(s) + 1000I(s) = -90e^{-s}.$$

Hence

$$I(s) = \frac{90(1-e^{-s})}{s^2 + 110s + 1000}.$$

This is precisely the same transform $I(s)$ we found in Example 5 of Section 7.5, so inversion of $I(s)$ yields the same solution $i(t)$ recorded there. ■

Example 3 Consider a mass on a spring with $m = k = 1$ and $x(0) = x'(0) = 0$. At each of the instants $t = 0, \pi, 2\pi, 3\pi, \ldots, n\pi, \ldots$, the mass is struck a hammer blow with a unit impulse. Determine the resulting motion.

Solution We need to solve the initial value problem

$$x'' + x = \sum_{n=0}^{\infty} \delta_{n\pi}(t); \quad x(0) = 0 = x'(0).$$

Because $\mathcal{L}\{\delta_{n\pi}(t)\} = e^{-n\pi s}$, the transformed equation is

$$s^2 X(s) + X(s) = \sum_{n=0}^{\infty} e^{-n\pi s},$$

so

$$X(s) = \sum_{n=0}^{\infty} \frac{e^{-n\pi s}}{s^2 + 1}.$$

We compute the inverse Laplace transform term by term; the result is

$$x(t) = \sum_{n=0}^{\infty} u(t - n\pi)\sin(t - n\pi).$$

Because $\sin(t - n\pi) = (-1)^n \sin t$ and $u(t - n\pi) = 0$ for $t < n\pi$, we see that if $n\pi < t < (n+1)\pi$, then

$$x(t) = \sin t - \sin t + \sin t - \cdots + (-1)^n \sin t;$$

that is,

$$x(t) = \begin{cases} \sin t & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

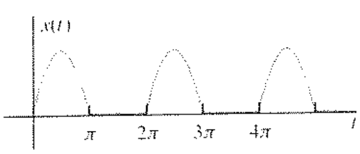


FIGURE 7.6.5. The half-wave rectification of $\sin t$.

Hence $x(t)$ is the half-wave rectification of $\sin t$ shown in Fig. 7.6.5. The physical explanation is that the first hammer blow (at time $t = 0$) starts the mass moving to the right; just as it returns to the origin, the second hammer blow stops it dead; it remains motionless until the third hammer blow starts it moving again, and so on. Of course, if the hammer blows are not perfectly synchronized then the motion of the mass will be quite different.

Systems Analysis and Duhamel's Principle

Consider a physical system in which the *output* or *response* $x(t)$ to the *input* function $f(t)$ is described by the differential equation

$$ax'' + bx' + cx = f(t), \tag{22}$$

where the constant coefficients a, b, and c are determined by the physical parameters of the system and are independent of $f(t)$. The mass-spring-dashpot system and the series RLC circuit are familiar examples of this general situation.

For simplicity we assume that the system is initially passive: $x(0) = x'(0) = 0$. Then the transform of Eq. (22) is

$$as^2X(s) + bsX(s) + cX(s) = F(s),$$

so

$$X(s) = \frac{F(s)}{as^2 + bs + c} = W(s)F(s). \tag{23}$$

The function

$$W(s) = \frac{1}{as^2 + bs + c} \tag{24}$$

is called the **transfer function** of the system. Thus the transform of the response to the input $f(t)$ is the product of $W(s)$ and the transform $F(s)$.

The function

$$w(t) = \mathcal{L}^{-1}\{W(s)\} \tag{25}$$

is called the **weight function** of the system. From Eq. (24) we see by convolution that

$$x(t) = \int_0^t w(\tau)f(t - \tau)\,d\tau. \tag{26}$$

This formula is **Duhamel's principle** for the system. What is important is that the weight function $w(t)$ is determined completely by the parameters of the system. Once $w(t)$ has been determined, the integral in (26) gives the response of the system to an arbitrary input function $f(t)$.

In principle—that is, via the convolution integral—Duhamel's principle reduces the problem of finding a system's outputs for all possible inputs to calculation of the single inverse Laplace transform in (25) that is needed to find its weight function. Hence, a computational analogue for a physical mass–spring–dashpot system described by (22) can be constructed in the form of a "black box" that is hard-wired to calculate (and then tabulate or graph, for instance) the response $x(t)$ given by (26) automatically whenever a desired force function $f(t)$ is input. In engineering practice, all manner of physical systems are "modeled" in this manner, so their behaviors can be studied without need for expensive or time-consuming experimentation.

Example 4 Consider a mass–spring–dashpot system (initially passive) that responds to the external force $f(t)$ in accord with the equation $x'' + 6x' + 10x = f(t)$. Then

$$W(s) = \frac{1}{s^2 + 6s + 10} = \frac{1}{(s + 3)^2 + 1},$$

so the weight function is $w(t) = e^{-3t}\sin t$. Then Duhamel's principle implies that the response $x(t)$ to the force $f(t)$ is

$$x(t) = \int_0^t e^{-3\tau}(\sin\tau)f(t - \tau)\,d\tau.$$

Note that

$$W(s) = \frac{1}{as^2 + bs + c} = \frac{\mathcal{L}\{\delta(t)\}}{as^2 + bs + c}.$$

Consequently, it follows from Eq. (23) that the weight function is simply the response of the system to the delta function input $\delta(t)$. For this reason $w(t)$ is sometimes called the **unit impulse response**. A response that is usually easier to measure in practice is the response $h(t)$ to the unit step function $u(t)$; $h(t)$ is the **unit step response.** Because $\mathcal{L}\{u(t)\} = 1/s$, we see from Eq. (23) that the transform of $h(t)$ is

$$H(s) = \frac{W(s)}{s}.$$

It follows from the formula for transforms of integrals that

$$h(t) = \int_0^t w(\tau)\,d\tau, \quad \text{so that} \quad w(t) = h'(t). \tag{27}$$

Thus the weight function, or unit impulse response, is the derivative of the unit step response. Substitution of (27) in Duhamel's principle gives

$$x(t) = \int_0^t h'(t)f(t-\tau)\,d\tau \tag{28}$$

for the response of the system to the input $f(t)$.

APPLICATIONS: To describe a typical application of Eq. (28), suppose that we are given a complex series circuit containing many inductors, resistors, and capacitors. Assume that its circuit equation is a linear equation of the form in (22), but with i in place of x. What if the coefficients a, b, and c are unknown, perhaps only because they are too difficult to compute? We would still want to know the current $i(t)$ corresponding to any input $f(t) = e'(t)$. We connect the circuit to a linearly increasing voltage $e(t) = t$, so that $f(t) = e'(t) = 1 = u(t)$, and measure the response $h(t)$ with an ammeter. We then compute the derivative $h'(t)$, either numerically or graphically. Then according to Eq. (28), the output current $i(t)$ corresponding to the input voltage $e(t)$ will be given by

$$i(t) = \int_0^t h'(\tau)e'(t-\tau)\,d\tau$$

(using the fact that $f(t) = e'(t)$).

HISTORICAL REMARK: In conclusion, we remark that around 1950, after engineers and physicists had been using delta functions widely and fruitfully for about 20 years without rigorous justification, the French mathematician Laurent Schwartz developed a rigorous mathematical theory of *generalized functions* that supplied the missing logical foundation for delta function techniques. Every piecewise continuous ordinary function is a generalized function, but the delta function is an example of a generalized function that is not an ordinary function.

7.6

Solve the initial value problems in Problems 1 through 8, and graph each solution function $x(t)$.

1. $x'' + 4x = \delta(t)$; $x(0) = x'(0) = 0$
2. $x'' + 4x = \delta(t) + \delta(t - \pi)$; $x(0) = x'(0) = 0$
3. $x'' + 4x' + 4x = 1 + \delta(t - 2)$; $x(0) = x'(0) = 0$
4. $x'' + 2x' + x = t + \delta(t)$; $x(0) = 0$, $x'(0) = 1$
5. $x'' + 2x' + 2x = 2\delta(t - \pi)$; $x(0) = x'(0) = 0$
6. $x'' + 9x = \delta(t - 3\pi) + \cos 3t$; $x(0) = x'(0) = 0$
7. $x'' + 4x' + 5x = \delta(t - \pi) + \delta(t - 2\pi)$; $x(0) = 0$, $x'(0) = 2$
8. $x'' + 2x' + x = \delta(t) - \delta(t - 2)$; $x(0) = x'(0) = 2$

Apply Duhamel's principle to write an integral formula for the solution of each initial value problem in Problems 9 through 12.

9. $x'' + 4x = f(t)$; $x(0) = x'(0) = 0$
10. $x'' + 6x' + 9x = f(t)$; $x(0) = x'(0) = 0$
11. $x'' + 6x' + 8x = f(t)$; $x(0) = x'(0) = 0$
12. $x'' + 4x' + 8x = f(t)$; $x(0) = x'(0) = 0$
13. This problem deals with a mass m, initially at rest at the origin, that receives an impulse p at time $t = 0$. **(a)** Find the solution $x_\epsilon(t)$ of the problem

$$mx'' = pd_{0,\epsilon}(t); \quad x(0) = x'(0) = 0.$$

(b) Show that $\lim_{\epsilon \to 0} x_\epsilon(t)$ agrees with the solution of the problem

$$mx'' = p\delta(t); \quad x(0) = x'(0) = 0.$$

(c) Show that $mv = p$ for $t > 0$ ($v = dx/dt$).

14. Verify that $u'(t - a) = \delta(t - a)$ by solving the problem

$$x' = \delta(t - a); \quad x(0) = 0$$

to obtain $x(t) = u(t - a)$.

15. This problem deals with a mass m on a spring (with constant k) that receives an impulse $p_0 = mv_0$ at time $t = 0$. Show that the initial value problems

$$mx'' + kx = 0; \quad x(0) = 0, \quad x'(0) = v_0$$

and

$$mx'' + kx = p_0\delta(t); \quad x(0) = 0, \quad x'(0) = 0$$

have the same solution. Thus the effect of $p_0\delta(t)$ is, indeed, to impart to the particle an initial momentum p_0.

16. This is a generalization of Problem 15. Show that the problems

$$ax'' + bx' + cx = f(t); \quad x(0) = 0, \quad x'(0) = v_0$$

and

$$ax'' + bx' + cx = f(t) + av_0\delta(t); \quad x(0) = x'(0) = 0$$

have the same solution for $t > 0$. Thus the effect of the term $av_0\delta(t)$ is to supply the initial condition $x'(0) = v_0$.

17. Consider an initially passive RC circuit (no inductance) with a battery supplying e_0 volts. **(a)** If the switch to the battery is closed at time $t = a$ and opened at time $t = b > a$ (and left open thereafter), show that the current in the circuit satisfies the initial value problem

$$Ri' + \frac{1}{C}i = e_0\delta(t - a) - e_0\delta(t - b); \quad i(0) = 0.$$

(b) Solve this problem if $R = 100\ \Omega$, $C = 10^{-4}$ F, $e_0 = 100$ V, $a = 1$ (s), and $b = 2$ (s). Show that $i(t) > 0$ if $1 < t < 2$ and that $i(t) < 0$ if $t > 2$.

18. Consider an initially passive LC circuit (no resistance) with a battery supplying e_0 volts. **(a)** If the switch is closed at time $t = 0$ and opened at time $t = a > 0$, show that the current in the circuit satisfies the initial value problem

$$Li'' + \frac{1}{C}i = e_0\delta(t) - e_0\delta(t - a);$$

$$i(0) = i'(0) = 0.$$

(b) If $L = 1$ H, $C = 10^{-2}$ F, $e_0 = 10$ V, and $a = \pi$ (s), show that

$$i(t) = \begin{cases} \sin 10t & \text{if } t < \pi, \\ 0 & \text{if } t > \pi. \end{cases}$$

Thus the current oscillates through five cycles and then stops abruptly when the switch is opened (Fig. 7.6.6).

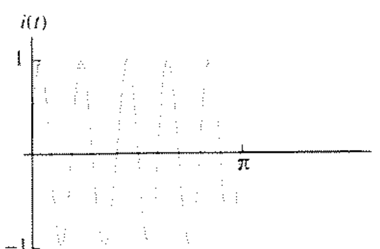


FIGURE 7.6.6. The current function of Problem 18.

19. Consider the LC circuit of Problem 18(b), except suppose that the switch is alternately closed and opened at times $t = 0, \pi/10, 2\pi/10, \ldots$. **(a)** Show that $i(t)$ satisfies the initial value problem

$$i'' + 100i = 10\sum_{n=0}^{\infty}(-1)^n\delta\left(t - \frac{n\pi}{10}\right); \quad i(0) = i'(0) = 0$$

(b) Solve this initial value problem to show that

$$i(t) = (n + 1)\sin 10t \quad \text{if} \quad \frac{n\pi}{10} < t < \frac{(n + 1)\pi}{10}.$$

Thus a resonance phenomenon occurs (see Fig. 7.6.7).

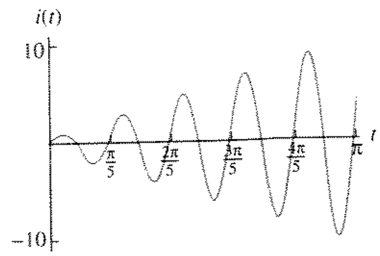


FIGURE 7.6.7. The current function of Problem 19.

Repeat Problem 19, except suppose that the switch is alternately closed and opened at times $t = 0, \pi/5, 2\pi/5, \ldots, n\pi/5, \ldots$. Now show that if

$$\frac{n\pi}{5} < t < \frac{(n+1)\pi}{5},$$

then

$$i(t) = \begin{cases} \sin 10t & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Thus the current in alternate cycles of length $\pi/5$ first executes a sine oscillation during one cycle, then is dormant during the next cycle, and so on (see Fig. 7.6.8).

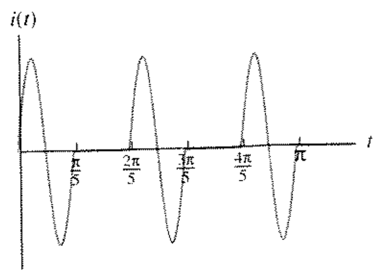


FIGURE 7.6.8. The current function of Problem 20.

21. Consider an *RLC* circuit in series with a battery, with $L = 1$ H, $R = 60\ \Omega$, $C = 10^{-3}$ F, and $e_0 = 10$ V. **(a)** Suppose that the switch is alternately closed and opened at times $t = 0, \pi/10, 2\pi/10, \ldots$. Show that $i(t)$ satisfies the initial value problem

$$i'' + 60i' + 1000i = 10\sum_{n=0}^{\infty}(-1)^n\delta\left(t - \frac{n\pi}{10}\right);$$

$$i(0) = i'(0) = 0.$$

(b) Solve this problem to show that if

$$\frac{n\pi}{10} < t < \frac{(n+1)\pi}{10},$$

then

$$i(t) = \frac{e^{3n\pi+3\pi} - 1}{e^{3\pi} - 1}e^{-30t}\sin 10t.$$

Construct a figure showing the graph of this current function.

22. Consider a mass $m = 1$ on a spring with constant $k = 1$, initially at rest, but struck with a hammer at each of the instants $t = 0, 2\pi, 4\pi, \ldots$. Suppose that each hammer blow imparts an impulse of $+1$. Show that the position function $x(t)$ of the mass satisfies the initial value problem

$$x'' + x = \sum_{n=0}^{\infty}\delta(t - 2n\pi); \quad x(0) = x'(0) = 0.$$

Solve this problem to show that if $2n\pi < t < 2(n+1)\pi$, then $x(t) = (n+1)\sin t$. Thus resonance occurs because the mass is struck each time it passes through the origin moving to the right—in contrast with Example 3, in which the mass was struck each time it returned to the origin. Finally, construct a figure showing the graph of this position function.