Math 2250-1
Tuesday Sept 9 §2.1-2.2

Recall, we were discussing the logistic population model

$$\frac{dP}{dt} = kP(M-P)$$ $k, M > 0$. M is "the carrying capacity" of the environment supporting the population

e.g. $\frac{dP}{dt} = 2P(3-P)$

isoclines = ?
predicted long time behavior with positive initial populations?

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analytic (formulas) solutions : separate variables!

$$\begin{cases} \frac{dP}{dt} = kP(M-P) \\ P(0) = P_0 \end{cases}$$

$$\int \frac{dP}{P(M-P)} = \int k\,dt$$

$$\frac{1}{M}\int \frac{1}{P} + \frac{1}{M-P}\,dP = kt + C \quad \left(\begin{matrix}\text{Partial}\\ \text{fractions!}\end{matrix}\right)$$

$$\frac{1}{M}\Big[\ln|P| - \ln|M-P|\Big] = kt + C$$

$$\frac{1}{M}\ln\left|\frac{P}{M-P}\right| = kt + C$$

$$\ln\left|\frac{P}{M-P}\right| = Mkt + \tilde{C}$$

$$\left|\frac{P}{M-P}\right| = e^{\tilde{c}}\,e^{Mkt}$$

$$\frac{P}{M-P} = Ce^{Mkt}$$

$P(0) = P_0$, so

$$\frac{P}{M-P} = \frac{P_0}{M-P_0}\,e^{Mkt}$$

$$P = \left(\frac{P_0}{M-P_0}\right)e^{Mkt}(M-P)$$

$$P\left[1 + \frac{P_0}{M-P_0}e^{Mkt}\right] = \frac{MP_0}{M-P_0}\,e^{Mkt}$$

$$\to P = \frac{\left(\frac{MP_0}{M-P_0}\right)e^{Mkt}}{1 + \left(\frac{P_0}{M-P_0}\right)e^{Mkt}}$$

so,!,

$$\boxed{P(t) = \frac{MP_0}{(M-P_0)e^{-Mkt} + P_0}}$$

long time behavior?
M is called the carrying capacity in logistic models. Why?

- Now discuss the U.S. population model - Maple notes from Monday

Before discussing more population (and also velocity-acceleration) models, let's talk about the general language that gets used...

§2.2 : A general 1st order DE $\frac{dx}{dt} = f(t, x)$

is called autonomous if $\frac{dx}{dt} = f(x)$ ($\frac{dx}{dt}$ only depends on x itself, not also on t)

def $x(t)$ is an equilibrium sol'n to a DE iff $x(t) \equiv C$, a constant

If the DE is autonomous and $x(t) \equiv C$ is an equilibrium soltn, then

$$0 = \frac{dx}{dt} = f(x) = f(C) = 0.$$

And if $f(C) = 0$ then $x(t) \equiv C$ is an equilibrium sol'n.

equilibrium sol's of $\frac{dx}{dt} = f(c)$ are exactly the fcns $x(t) \equiv c$ where $f(c) = 0$

example

$$\frac{dx}{dt} = kx(M - x)$$

$x \equiv 0$
$x \equiv M$ are the equil. sol'ns

example find the equil. sol'ns of

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

Def Let c be an equilibrium sol'n for a deqtn. Then

c is **stable** iff $\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t. for sol'ns $x(t)$ with $|x(0)-c| < \delta$ we have $|x(t)-c| < \varepsilon\ \ \forall\ t > 0$

(sol'ns which start close enough to c stay arbitrarily close to it.)

c is **unstable** if it is not stable

example: Make the slope field (or alternately, sketch a sufficient sample of sol'n graphs) and phase portrait for

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

and discuss stability of the equil sol'ns $x=0, x=-1$

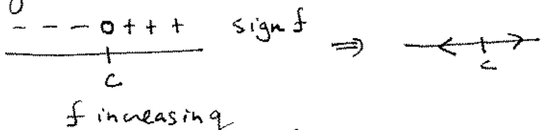
example find equilibria and discuss stability for $\frac{dx}{dt} = 3x - x^2$

Def c is called **asymptotically stable** (stronger requirement than stable) iff $\exists\ \delta > 0$ s.t. $|x(t)-c| < \delta \Rightarrow \lim_{t\to\infty} x(t) = c$.

Are any of the equilibria you found above asymptotically stable?

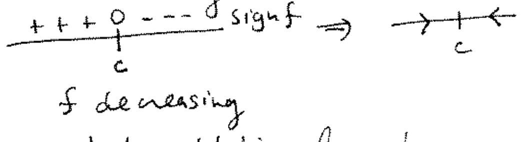
Theorem: Consider $\frac{dx}{dt} = f(x)$, f continuously differentiable.

if $f(c) = 0$ then $f'(c) > 0 \Rightarrow$



Unstable

$f'(c) < 0 \Rightarrow$



stable

$f'(c) = 0$ You must do additional work.

More "real" examples

doomsday extinction:

$$\frac{dP}{dt} = aP^2 - bP \qquad a,b>0.$$

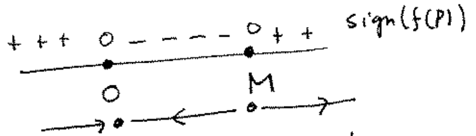
$$= kP(P-M) \qquad \begin{matrix} k=a \\ -kM=b \end{matrix}$$

e.g. if $\beta(t)$ (fertility rate) $\overset{\text{proportional to}}{\downarrow} \sim P$

(e.g. herd roams in a fixed area and mates randomly upon meeting opposite-sex mate)

then $B(t) = aP^2$;

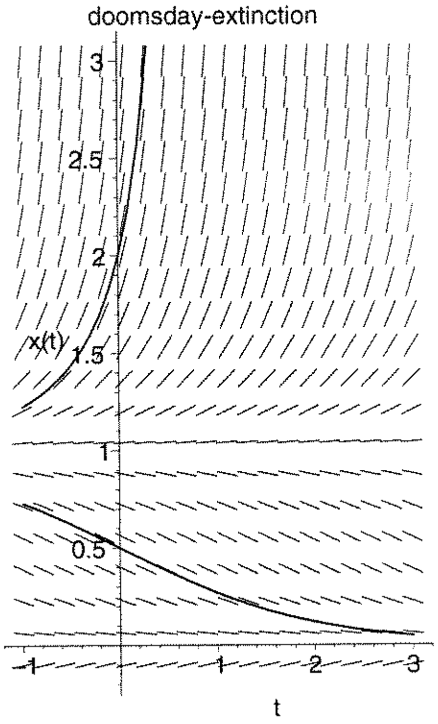
perhaps $D(t) = -bP$.



0 stable; M Unstable!

If $0 < P_0 < M$ then $\lim_{t\to\infty} P(t) = 0$ (so in real life, extinction)

If $P_0 > M$ then $\exists\, t_1 < \infty$ s.t. $\lim_{t\to t_1} P(t) = +\infty$ (doomsday!)


doomsday-extinction

example

$$\begin{cases} \dfrac{dx}{dt} = x(x-1) \\ x(0) = 2 \end{cases}$$

Find the time of doomsday.

(ans $t = \ln 2$!)