Math 2250-1
Tuesday Sept 30

Finish determinants,
and nifty formulas for A^{-1}
and Cramer's rule for solving $A\vec{x}=\vec{b}$.

Recall recursive def of det:

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij} \quad (\text{expansion accross } row_i(A))$$

$$= \sum_{i=1}^{n} a_{ij}C_{ij} \quad (\text{expansion down } col_j(A)).$$

(book writes A_{ij}) ↓

$$C_{ij} = \underset{\substack{\uparrow \\ \text{precisely} \\ (-1)^{i+j}}}{\pm} M_{ij}$$

"Minor" precisely the $(n-1)\times(n-1)$ det of matrix obtained from A by deleting $row_i(A)$ & $col_j(A)$.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & & \\ + & & \ddots & \\ \vdots & & & \end{bmatrix}$$

- expanding down <u>any</u> column, or across <u>any</u> row yields same value, i.e. $|A|$.
- If A is (upper or lower) triangular, $|A|$ is the product of its diagonal terms.

Effects of elementary row operations (<u>or column ops</u>) on determinants:

(1) swapping two rows (or two columns) changes sign of determinant

proof: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$; $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc-ad$, so true for 2×2.

$n=3 \rightarrow$ expand across the row that <u>wasn't</u> swapped, all the 2×2 minors change sign, so the det does too!

In general, the proof follows by <u>math induction</u>: if result is true for all n×n matrices, then deduce it for $A_{(n+1)\times(n+1)}$ by expanding along any unswapped row ∎

(1b): So, if 2 rows are equal, $|A|=0$!

reason: Let $x=|A|$. If you swap the identical rows the new matrix has det $= -x$. But the new matrix is the old matrix, so $x=-x$, i.e. $2x=0$, $x=0$ ∎

(2) multiplying a single row by a constant multiplies det by same const:

$$\begin{vmatrix} R_1 \\ R_2 \\ \vdots \\ cR_i \\ \vdots \\ R_n \end{vmatrix} \underset{\substack{\uparrow \\ \text{expand} \\ \text{across} \\ row_i}}{=} \sum_{j=1}^{n} (ca_{ij})\underset{\substack{\uparrow \\ \text{still } A\text{-cofactors!}}}{C_{ij}} = c\sum_{j=1}^{n} a_{ij}C_{ij} = c|A|$$

∎

(the effect of this is that when we do row ops, if we factor a "c" out of a row we multiply it by the det of what's left to get original det.)

(3) replace $\text{row}_i(A)$ with $\text{row}_i(A) + c\,\text{row}_k(A)$:

Does not change det!!!

reason:

$$\begin{array}{r} \\ \\ \text{row}_i \rightarrow \\ \\ \\ \end{array} \begin{vmatrix} R_1 \\ R_2 \\ \vdots \\ R_i + cR_k \\ \vdots \\ R_n \end{vmatrix} \underset{\substack{\uparrow \\ \text{expand} \\ \text{across row}_i}}{=} \sum_{j=1}^{n} (a_{ij} + c a_{kj}) \underset{\substack{\uparrow \\ \text{still } A \text{ cofactors!}}}{C_{ij}} = \underbrace{\sum_{j=1}^{n} a_{ij} C_{ij}}_{\substack{\| \\ |A|}} + c \underbrace{\sum_{j=1}^{n} a_{kj} C_{ij}}_{\substack{\| \\ \cancelto{0}{\begin{vmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \\ \vdots \\ R_k \\ \vdots \\ R_n \end{vmatrix}} \begin{array}{l} \\ \\ \\ \leftarrow \text{row}_k \\ \\ \leftarrow \text{row}_i \\ \\ \\ \end{array}}}$$

∎

<u>example</u>

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$\begin{array}{r} \\ \\ -2R_1+R_3 \end{array} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix}$$

$$\begin{array}{r} \\ \\ +2R_2+R_3 \end{array} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

$$\begin{array}{r} \\ R_2/3 \\ R_3/5 \end{array} = 3 \cdot 5 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 3 \cdot 5 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ (various cleanup row ops, all of "no-change" type)

$$= 15 \cdot 1 = 15.$$

③

Theorem $\det A \neq 0$ iff $\text{rref}(A) = I$ iff A^{-1} exist

already know this equivalence

pf: start with A

do elementary row ops

- change sign if swap rows (original det is opposite of new det.)
- no change if replace row_i by $\text{row}_i + c\,\text{row}_k$ $k \neq i$
- if factor "c" out of row, original det is c times det of what's left

rref(A)

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if $\text{rref}(A) = I$

$$|A| = \underbrace{c_1 c_2 \cdots c_N}\ \underbrace{|I|}_{1}$$

c_i's are ±1, or other nonzero const's.

and $|A| = c_1 c_2 \cdots c_N \neq 0$!

if $\text{rref}(A) \neq I$

rref(A); has a row of 0's.

$$|A| = c_1 c_2 \cdots c_N \begin{vmatrix} \cdots \\ 0\,0\,0 \cdots 0 \end{vmatrix} = 0$$

∎

Thm $\det(AB) = (\det A)(\det B)$ ← true, but not obvious:

$\det(A+B) \neq \det A + \det B$!!!

$|A|$

rref

$\text{rref}(A) = I$

$$|A| = \underbrace{c_1 \cdots c_N}_{|A|} |I|$$

$\text{rref}(A) \neq I$

$$c_1 \cdots c_N \begin{vmatrix} \cdots \\ 0\,0\,0 \end{vmatrix}$$

$|AB|$

do indentical rowops to $AB \to$ is same as doing row ops to A, then multiplying

$\text{rref}(A) = I$

$$|AB| = c_1 \cdots c_N |IB| = |A||B|$$

$\text{rref}(A) \neq I$

$$|AB| = c_1 \cdots c_N \left| \begin{pmatrix} \cdots \\ 0\,0\,0 \end{pmatrix} B \right| = c_1 \cdots c_N \begin{vmatrix} \cdots \\ 0\,0\,0\,0 \end{vmatrix} = 0 = |A||B|.$$

If $B_{m\times n} = [b_{ij}]$ then the <u>transpose</u> of B, B^T is defined by

$$\text{entry}_{ij}(B^T) := \text{entry}_{ji}(B) = b_{ji}$$

the effect is to switch rows & columns (and vise verse)

e.g. ~~ex.~~ $\text{entry}_{ij}(B^T) := \text{entry}_j(\text{row}_i(B^T))$

$$\parallel$$

$$b_{ji} = \text{entry}_j(\text{col}_i(B))$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

<u>Definition</u> the adjoint matrix of $A = [a_{ij}]$ (write $\text{adj}(A)$) is the transpose of the cofactor matrix $[C_{ij}]$ $\qquad \text{adj}(A) = [C_{ij}]^T$

<u>Theorem</u> When A^{-1} exists, $\boxed{A^{-1} = \frac{1}{|A|}\,\text{adj}(A)}$

<u>examples</u>

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad [M_{ij}] = \begin{bmatrix} d & c \\ b & a \end{bmatrix}; \quad [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}; \quad \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

so $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$!

our friend

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$[C_{ij}] = \begin{bmatrix} \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

check!

proof that $A^{-1} = \underbrace{\frac{1}{|A|}\,\mathrm{adj}(A)}_{X}$:

does $AX = I$?

$$\mathrm{entry}_{\ell m}(AX) = \sum_{k=1}^{n} a_{\ell k}\, \underbrace{X_{km}}_{\frac{1}{|A|}C_{mk}}$$

$$= \frac{1}{|A|} \sum_{k=1}^{n} a_{\ell k}\, C_{mk}$$

→ if $\ell = m$ this sum is expansion for $|A|$ down $\ell = m^{th}$ col

so $\mathrm{entry}_{mm}(AX) = \frac{|A|}{|A|} = 1$

↘ if $\ell \neq m$ we are expanding a det. down the m^{th} col of a matrix in which we replaced column m with the ℓ^{th} column, so two col's are equal, so det $= 0$!

this shows $AX = I$. ▨

(which implies $XA = I$).

Cramer's rule

Let $\vec{x}$ solve $A\vec{x} = \vec{b}$ for invertible A.

then $x_k = \dfrac{\left| C_1 \middle| C_2 \middle| \cdots \middle| b \middle| \cdots C_n \right|}{|A|}$

numerator is det of matrix obtained from A by replacing ~~row~~ column k by $\vec{b}$.

proof: $x_k = \mathrm{entry}_k(A^{-1}\vec{b})$

$= \mathrm{entry}_k\left(\frac{1}{|A|}\,\mathrm{adj}(A)\vec{b}\right)$

$= \frac{1}{|A|}\,\mathrm{row}_k(\mathrm{adj}(A))\cdot\vec{b}$

$= \frac{1}{|A|} \underbrace{\sum_{\ell=1}^{n} C_{\ell k}\, b_\ell}$

this is the expansion of the det in numerator of Cramer's rule, down the k^{th} column!

▨