Math 2250-1
Wednesday Sept. 3

Maple Intro sessions
    Wed (today) 10:45-11:35 (LCB 115)
    Thurs (tomorrow) 9-10, 10:40-11:40 (MMC PCLab 1735)
  do we need more?

- Derive Toricelli's Law for draining cisterns,
  and the separable DE it implies

  - this is page 3 Tuesday notes.
    there are a couple of sign errors on that page, which we will correct,
    as we go through the notes. The final boxed DE is correct

- If $T_\mu$ is the time it takes for water to drain from height $h$ to height $\mu h$ <u>in a cylinder</u>

  show
  $$T_0 = \frac{T_\mu}{1 - \sqrt{\mu}} \qquad (\text{page 4 Tues.})$$

- In our experiment yesterday

  $$T_{\frac{1}{2}} = 34.2 \text{ sec.}$$
  $$T_0 = 108 \text{ sec.}$$

  Use the value of $T_{\frac{1}{2}}$ to see what our model predicts for $T_0$.

  What was our <u>relative error</u>
  Can you think of measurement & model errors which could contribute to this?

§1.5 Linear $1^{st}$ order DE's.

(1) $\qquad \dfrac{dy}{dx} + P(x)y = Q(x)$

Notice the left side of this eqtn, $L(y) := y' + P(x)y$ is <u>linear</u> :

$$L(y_1 + y_2) = L(y_1) + L(y_2) \qquad (y_1, y_2 \text{ diffble})$$
$$L(cy) = cL(y) \qquad (c \text{ a const, } y \text{ diffble})$$

Solution method is to multiply both sides by a non-zero fcn ("integrating factor") so that we can antidifferentiate wrt $x$ to deduce $y(x)$:

eqtn (1) is equiv. to

(2) $\qquad e^{\int P(x)dx}\left(y' + P(x)y\right) = e^{\int P(x)dx} Q(x)$

where $\int P(x)dx$ is any particular antideriv. of $P(x)$

equiv to

(3) $\qquad \dfrac{d}{dx}\left[ e^{\int P(x)dx} y \right] = \underbrace{e^{\int P(x)dx} Q(x)}$

or (antidifferentiating)

(4) $\qquad e^{\int P(x)dx} y = \int [\quad] dx$ ; divide by $e^{\int P(x)dx}$ to get $y(x)$

<u>Example</u> HW problem #6 §1.3 you are asked to plug in $y = x + Ce^{-x}$ to show it solves the DE

$$y' = x - y + 1$$

Use the algorithm above to <u>derive</u> the given solns :
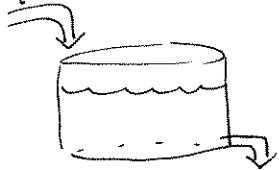
$$y' + 1y = x + 1$$

example 2   page 48-49:

$$(x^2+1) \frac{dy}{dx} + 3xy = 6x$$

ans  $y = 2 + C(x^2+1)^{-3/2}$

• Example: <u>mixing problems</u> (p. 51)   applies also to pharmacology & environment

$r_i$ = rate in   $\ell/s$
$c_i$ = concentration in    $gm/\ell$



$r_0$ = rate out $\ell/s$
$c_0$ = concentration out $g/\ell$

$V(t)$ = volume in tank at time $t$   $(\ell)$
$x(t)$ = amount of solute in tank   $(gm)$

$c(t) = \dfrac{x(t)}{V(t)}$   $\dfrac{g}{\ell}$   (average) concentration in tank

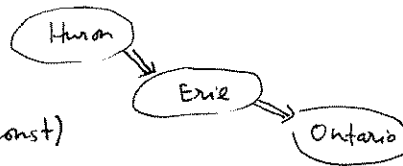Assume mixture is uniform so $c(t)$ is spatially constant

$$\dfrac{dV}{dt} = r_i - r_0 \qquad \ell/s \longrightarrow V(t) = V_0 + \int_0^t r_i(s) - r_0(s)\, ds \qquad \S 1.2$$

$$\dfrac{dx}{dt} = r_i c_i - r_0 c_0 \qquad \dfrac{\ell}{s}\dfrac{g}{\ell}\ g/s$$

$$= r_i c_i - r_0 \dfrac{x(t)}{V(t)} \qquad \text{by "well mixed" model}$$

$$\boxed{\dfrac{dx}{dt} + \dfrac{r_0}{V(t)} x(t) = r_i c_i}$$

$\uparrow$  $\uparrow$
$P(t)$  $Q(t)$

<u>Example 4 p. 52</u>   Lake Erie pollution



$V(t) = 480\ km^3$
$r = r_i = r_0 = 350\ km^3/year$ (so V is const)
$c_i = c$ = pollutant conc of lake Huron
$x_0 = (5c)V$   (read problem).

When will Erie pollution concentration have decreased to twice that of Huron?

i.e. solve, and analyze

$$\begin{cases} \dfrac{dx}{dt} = r_i c_i - r_o c_o \\[3mm] x(0) = 5cV \end{cases}$$

$$\dfrac{dx}{dt} = rc - r\dfrac{x}{V}$$

for fun, notice this DE is also separable. Can you solve it that way?

$$\dfrac{dx}{dt} + \dfrac{r}{V} x = rc$$

$$\left(e^{\frac{r}{V}t} x\right)' = rc\, e^{\frac{r}{V}t}$$

$$e^{\frac{r}{V}t} x = rc\, \dfrac{V}{r} e^{\frac{r}{V}t} + C$$

$$x = cV + Ce^{-\frac{r}{V}t}$$

$$x(0) = 5cV \implies C = 4cV$$

$$x(t) = cV + 4cV\, e^{-\frac{r}{V}t}$$

set $\quad x(T) = 2cV = cV + 4cV e^{-\frac{r}{V}T}$

$$.25 = e^{-\frac{r}{V}T}$$

$$-\dfrac{r}{V}T = \ln(.25)$$

$$T = -\ln(.25)\dfrac{V}{r}$$

$$= \ln(4)\dfrac{480}{350} \quad \text{years}$$

$$\boxed{\approx 1.90 \text{ years}}$$