Math 2250-1
Monday Sept 29

Determinants §3.6

The determinant of a square matrix is a magic number, obtained by strange rules
The matrix is invertible exactly when $\det(A) \neq 0$
exactly when $\operatorname{rref}(A) = I$

When $\det A \neq 0$ there is an algebraic formula for A^{-1}; (adjoint formula)
which results in a determinant formula (Cramer's rule)
for finding solutions to $A\vec{x} = \vec{b}$

Illustrate for 2×2 matrices :

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

bars rather than brackets
mean take determinant.

if $\det A \neq 0$ then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ check!

example: solve (a 2×2 system where the coefficients haven't been chosen to be "nice")

$$\begin{aligned} 3x + 7y &= 5 \\ 5x + 4y &= 8 \end{aligned}$$

determinant can be defined inductively (i.e. for matrices $(n+1)\times(n+1)$ if you know $n\times n$ dets)

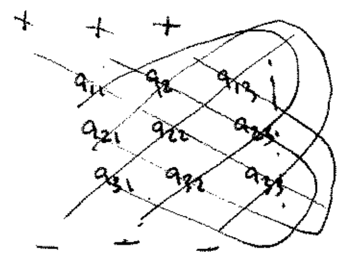
$\boxed{3\times 3 \text{ case:}}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{M_{11}} \overset{(-1)^{1+2}}{\downarrow}\!\!- a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{M_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{M_{13}}$$

M_{11} (1-1 minor) cross out row$_1$ & col$_1$, take 2×2 det

$$= a_{11}(a_{22}a_{33}-a_{32}a_{23}) - a_{12}(a_{21}a_{33}-a_{31}a_{23}) + a_{13}(a_{21}a_{32}-a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$



"cross hatch" only works for 3×3!

example:

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$=$

in fact, using matrix of $(-1)^{i+j} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

you can compute a 3×3 det by expanding across any row or down any column:

$$\det A = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} \overset{ij \text{ minor}}{M_{ij}} = \sum_{j=1}^{3} a_{ij} \overset{ij \text{ cofactor}}{C_{ij}} \quad [i \text{ fixed}$$

expansion across row i

$$= \sum_{i=1}^{3} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{3} a_{ij} C_{ij} \quad [j \text{ fixed}$$

down column j

recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$

by expanding down column 1
and by cross-hatching.
Get same answer!

Inductive def'n of det A:
recursive)

$$\det A := \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} C_{1j}$$

(expansion across top row)

("ij Minor")
$M_{ij} :=$ $(n-1)\times(n-1)$ det of matrix obtained from A by deleting $\text{row}_i(A)$ & $\text{col}_j(A)$
$C_{ij} := (-1)^{i+j} M_{ij}$
("ij cofactor")

<u>Theorem</u> (true, but not easy, see appendix for details)

You can compute det A by expanding across any row or down any column:

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{(fixed } i\text{)}$$

$$= \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{(fixed } j\text{)}$$

Example

$$\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 96 & \pi & 3 & 0 \\ 1221 & 34 & 17 & 4 \end{vmatrix}$$

Theorem: If A is upper or lower triangular then $|A|$ is just the product of the diagonal entries

$\boxed{\text{Computational Shortcuts:}}$ (and important for understanding too!)

Effects of elementary row ops on determinants:

(analogous results hold for elementary column operations)

(1) swapping two rows changes the sign of the determinant

proof: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$; $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$

so true for 2×2.

Now it is easy to check general case by <u>induction</u>

$n=3 \rightarrow$ expand across the row that wasn't swapped and use the $n=2$ result on each minor.

In general, for $A_{(n+1)\times(n+1)}$ expand across an unswapped row and use the (inductive) assumption that result is true for $n \times n$ matrices.

(1b) So, if 2 rows are equal, $\det = 0$

<u>proof</u>: Let $\det(A) = x$

if we swap rows the new det is $-x$

but rows were the same, so had to get same det,

i.e. $-x = x \Rightarrow x = 0$.

(2) multiplying a <u>single</u> row by c, multiplies the det by c.

proof: if you multiplied $\text{row}_i(A)$ by c, expand new matrix det across row_i:

$$\begin{vmatrix} R_1 \\ R_2 \\ cR_i \\ \vdots \\ R_n \end{vmatrix} = \sum_{j=1}^{n} (c a_{ij}) C_{ij} = c \sum_{j=1}^{n} a_{ij} C_{ij} = c \det A.$$

(3) Coolest property: replacing $\text{row}_i(A)$ with $\text{row}_i(A) + c\,\text{row}_k(A)$ Does Not change det!

proof: we'll expand across $\text{row}_i(A)$:

$$\begin{array}{r} \\ \text{row}_i \rightarrow \\ \\ \end{array} \begin{vmatrix} R_1 \\ R_2 \\ R_i + cR_k \\ \vdots \end{vmatrix} = \sum_{j=1}^{n} (a_{ij} + c a_{kj}) C_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_j$$

$$= \det(A) + c \underbrace{\begin{vmatrix} R_1 \\ R_2 \\ R_k \\ \vdots \\ R_k \\ R_n \end{vmatrix}}_{\rightarrow 0 \text{ by (1b)}} \begin{array}{l} \\ \\ \leftarrow k^{th} \text{ row} \\ \\ \leftarrow i^{th} \text{ row} \\ \\ \end{array}$$

example

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \begin{array}{r} \\ \\ -2R_1+R_3 \end{array} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix}$$

$$= \begin{array}{r} \\ \\ 2R_2+R_3 \end{array} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 15 \quad !$$

We'll continue this discussion Tuesday, figure out that A^{-1} exists exactly when $\det A \neq 0$, and find the magic formula for the inverse in this case.